On optimization of dividend flow for a company with nonnegative liquidation value

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Abstract

We consider a model for a firm whose reserve X_t evolves according to

$$dX_t = \mu dt + \sigma dW_t - dZ_t, \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process and μ and σ are positive constants. The control functional Z_t represents the cumulative amount of dividends paid-out up to time t. The major requirements on the control process Z_t is that it is nonnegative, nondecreasing and adapted to the filtration. The bankruptcy time τ is defined as $\tau = \inf(t \geq 0 : X_t \leq 0)$. It is assumed that the initial reserve $X_0 = x$ is positive and that the liquidation value P, i.e. the salvage value of the firm's assets at the time of bankruptcy, is nonnegative. With λ the constant discount rate, the expected total amount of dividends paid out during the existence of the firm equals

$$V(x, Z_t) = \mathbf{E}_x\{\int_0^\tau e^{-\lambda t} dZ_t + Pe^{-\lambda\tau}\}. \tag{2}$$

We will present explicit formulae for the optimal admissible process Z, i.e. the control process that maximizes $V(x, Z)$, both for the case of bounded dividend rates and for the case of unbounded dividend rates.

Contents

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1 Introduction

1.1 Choice of reserve process and structure of dividend process

Radner and Shepp (1996) [4] have proposed a model for a firm whose reserve X_t in absence of dividends evolves as an arithmetic Brownian motion[1]

$$dX_t = \mu dt + \sigma dW_t, \tag{3}$$

here μ and σ are some positive constants.

The firm's manager influences the stochastic fluctuations of the company's reserve by choosing the time and the size of dividend payments. The dynamics of the reserve become

$$dX_t = \mu dt + \sigma dW_t - dZ_t, \tag{4}$$

where the process $Z = (Z_t)_{t\geq 0}$ represents the cumulative amount of dividends paid out up to time t. Z_t is considered to be a nonnegative, non-decreasing, right-continuous functional.

First we consider a dividend process with bounded dividend rates. In other words the rate of dividend payments is bounded by a given positive constant K in this case . The dividend process Z_t is given by

$$Z_t = Z_0 + \int_0^t u(X_s)ds, \tag{5}$$

where Z_0 is an initial dividend payment, Z_0 does not exceed the initial reserve $X_0 = x$, i.e. $0 \leq Z_0 \leq x$,
u is the bounded dividend rate, $0 \leq u(x) \leq K < \infty$,
$u = u(x), Z_0 = Z_0(x)$ are assumed to be measurable.

As a limiting case for $K \to \infty$ we consider a dividend process with unbounded dividend rates. In this case Z_t is assumed to be any nonnegative non-decreasing right-continuous functional.

The processes Z_t defined above are called *admissible controls*.

The firm exist from time zero until the first moment τ, at which the cash reserve falls down to zero ($\tau = \inf\{t : X_t \leq 0\}$). The moment τ is called *the bankruptcy time*.

1.2 The value function

The aim of the manager is to maximize the expected total discounted dividends paid out during the existence of the firm assuming the firm receives positive salvage value at the time of bankruptcy. In other words the value function for our optimization problem is given by

$$V(x) = \sup \mathbf{E}_x\{\int_0^\tau e^{-\lambda t} dZ_t + Pe^{-\lambda\tau}\}, \tag{6}$$

where the mathematical expectation is taken with respect to initial reserve x,
$P = const \geq 0$ is the liquidation value, i.e. the salvage value of the firm's assets at the time of bankruptcy,
$\lambda = const > 0$ is the discount,
and sup is taken over all admissible processes $Z = (Z_t)_{t\geq 0}$.

[1] They also showed why geometric Brownian motion is an inappropriate model for the reserve.

1.3 Objective

Our aim is to find the value function $V(x)$ and the optimal dividend policy $\tilde{Z}_t(x)$ given an initial reserve x.

1.4 Normalization

Without loss of generality we assume $\sigma^2 = 1$ and change

$$\mu \to \frac{\mu}{\sigma^2}, \lambda \to \frac{\lambda}{\sigma^2}, K \to \frac{K}{\sigma^2}. \tag{7}$$

1.5 Overview

A survey on the subject may be found in Taksar (1999) [5]. The case of zero liquidation value $P = 0$ was solved by Jeanblanc and Shiryaev (1995) [3]. Here we present the general solution for the case $P \geq 0$.

2 The main result

2.1 The case of bounded dividend rate

Suppose we are given initial reserve $x \geq 0$, liquidation value $P \geq 0$, discount rate $\lambda > 0$, and an upper bound on dividend rate $u(x)$ K. Let us denote

$$r_1 = -\mu + \sqrt{\mu^2 + 2\lambda} > 0, \tag{8}$$

$$r_2 = -\mu - \sqrt{\mu^2 + 2\lambda} < 0, \tag{9}$$

$$\rho = K - \mu - \sqrt{(K-\mu)^2 + 2\lambda} < 0, \tag{10}$$

$$Q = \frac{K}{\lambda} + \frac{1}{K - \mu - \sqrt{(K-\mu)^2 + 2\lambda}}. \tag{11}$$

Define the functions $u_1(\cdot)$, $u_2(\cdot)$ by

$$u_1(x) = \frac{r_1 e^{r_1 x} - r_2 e^{r_2 x}}{r_1 - r_2}, \ x \geq 0, \tag{12}$$

$$u_2(x) = \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}, \ x \geq 0. \tag{13}$$

$$\tag{14}$$

Also let $\tilde{x}$ be the unique positive solution of the equation

$$\frac{1 - Qr_2}{r_1 - r_2} e^{-r_1 x} - \frac{1 - Qr_1}{r_1 - r_2} e^{-r_2 x} = P \tag{15}$$

The the optimal control process in case of bounded dividend rate is of the bang-bang type and may be described now as follows

Theorem 2.1.1 *The value function $V = V(x)$ and the optimal dividend strategy in the model for bounded dividend rates are the following*

1) if $P \geq Q$ then the optimal strategy is to distribute the reserve $X_0 = x$ as dividends immediately, so $Z_0 = x$, and liquidate the company at instant, so $X_t = 0$ for all $t > 0$. The value function is $V(x) = P + x$.

2) if $P < Q$, then the optimal instantaneous dividend payment is

$$Z_0(x) = \begin{cases} 0, & x < \tilde{x} \\ x - \tilde{x}, & x \geq \tilde{x} \end{cases},$$

the optimal dividend rate is given by

$$\tilde{u}(x) = \begin{cases} 0, & x < \tilde{x} \\ K, & x \geq \tilde{x} \end{cases},$$

and the value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < \tilde{x} \\ Q + x - \tilde{x}, & x \geq \tilde{x} \end{cases},$$

with

$$A = \frac{1 - Pu_1'(\tilde{x})}{u_1(\tilde{x})},$$

where $\tilde{x}$ is given by (15).

This theorem means that whenever the reserve x hits a certain value $\tilde{x}$, one should start distributing the dividends at the maximum possible rate K until the reserve falls back below $\tilde{x}$. This is illustrated in Fig. 1.

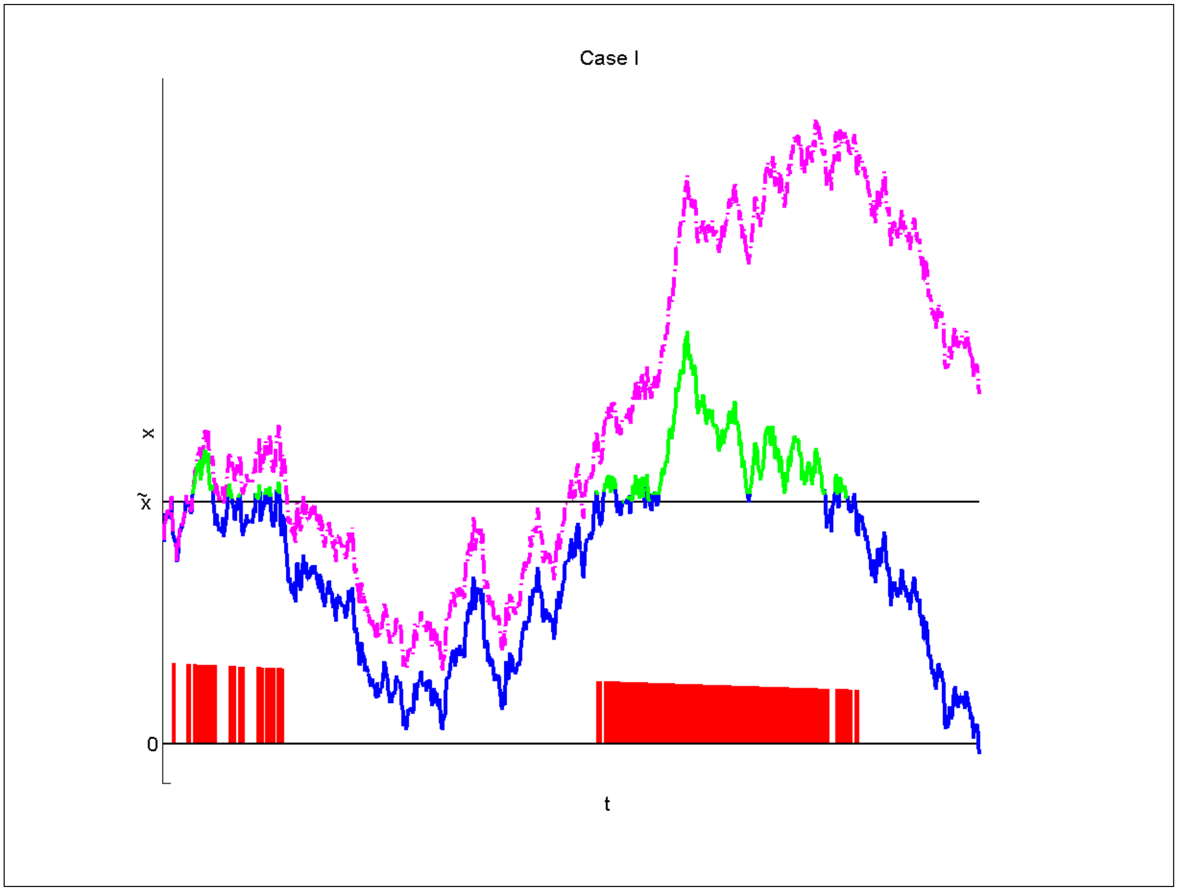

Figure 1. An example of reserve trajectory and dividend payments for the case of bounded dividend rate.

dashed magenta: an uncontrolled trajectory

blue: reserve when the control is off

green: reserve when the control is on

red: discounted dividends paid.

Note 2.1.1 *In the important special case* $P = 0$ *the statements* $P < Q$ *and* $P \geq Q$ *are equivalent by lemma 3.1.1 to* $K > \lambda/(2\mu)$ *and* $K \leq \lambda/(2\mu)$ *respectively. Returning to the original normalization these inequalities become* $K < \frac{\sigma^2}{2}\frac{\lambda}{\mu}$ *and* $K \geq \frac{\sigma^2}{2}\frac{\lambda}{\mu}$. *The latter inequalities might have nice economic explanations.*

2.2 The case of unbounded dividend rate

Suppose again we are given initial reserve $x$and liquidation value $P \geq 0$, but assume now that the dividend rate is not bounded. This case can be obtained from the discussed above by allowing the bound K on the dividend rate to go to infinity, $K \to \infty$. Let $r_1, r_2, u_1(\cdot), u_2(\cdot)$ be defined as above by (8), (9), (12), (13) respectively. Define $\bar{x}$ as the unique positive solution of the equation

$$\frac{r_2^2}{r_1 - r_2}e^{-r_1 x} - \frac{r_1^2}{r_1 - r_2}e^{-r_2 x} = P\lambda. \tag{16}$$

The following theorem presents the optimal strategy for this case

Theorem 2.2.1 *1) if* $P \geq \frac{\mu}{\lambda}$ *then the optimal strategy will be to distribute the reserve* $X_0 = x$ *as dividends immediately, so* $Z_0 = x$, *and hence to go bankrupt at the start, so* $X_t = 0$ *for all* $t > 0$. *This strategy yields value function* $V(x) = P + x$.

2) if $P < \frac{\mu}{\lambda}$ *then*

a) if $x > \bar{x}$ *dividends* $Z_0(x) = x - \bar{x}$ *should be paid*

b) if $x = \bar{x}$ *then the process of dividend payments* $\bar{L} = (\bar{L}_t)_{t \geq 0}$ *and the reserve process* $\bar{X} = (\bar{X}_t)_{t \geq 0}$ *are the solutions of a SDE with reflection* $\bar{X}_t = x + \mu t + \sigma W_t - \bar{L}_t$, *where* $\bar{L} = (\bar{L}_t)_{t \geq 0}$ *is a continuous nondecreasing and adapted process with* $\bar{L}_0 = 0$ *such that* $\bar{L}_t = \int_0^t I(\bar{X}_s = \bar{x})d\bar{L}_s$.

The value function is given by

$$V(x) \quad = \quad \left\{ \begin{array}{cc} Pu_1(x) + Au_2(x), & x < \bar{x} \\ x - \bar{x} + \frac{\mu}{\lambda}, & x \geq \bar{x} \end{array} \right\},,$$

with

$$A = \frac{1}{u_1(\bar{x})}[1 - Pu_1'(\bar{x})], \tag{17}$$

where $\bar{x}$ *is given by (16).*

This means[2] that whenever the reserve x hits a certain value $\bar{x}$ one should start distributing the dividends so fast that the reserve reflects at $\bar{x}$ and the dividends come from the local time spent by the reserve at $\bar{x}$.

3 Obtaining a solution in the case of bounded dividend rate

In this case the reserve of the company is given by

$$X_t = (x - Z_0) + \int_0^t (\mu - u(X_s))ds + W_t, \tag{18}$$

[2]remember that this is the limiting case for bounded dividend rate as $K \to \infty$.

where Z_0 is an initial dividend payment, x is the initial reserve, and u is the dividend rate.

Let $P \geq 0$ be the liquidation value. We measure optimality of dividend policy by the functional of the first dividend payment Z_0 and of continuous dividend yield policy u

$$\begin{aligned} V(x,u,Z_0) &= Z_0(x) + E_{x-Z_0(x)}\{\int_0^\tau e^{-\lambda t}u(X_t)dt + Pe^{-\lambda\tau}\} \\ V(x) &= \sup_{0\leq u(x)\leq K, 0\leq Z_0(x)\leq x} V(x,u,Z_0). \end{aligned} \tag{19}$$

It is obvious from (19) that

$$V(x) = \max_{0\leq Z_0\leq x}\{Z_0 + V_0(x - Z_0(x))\}, \tag{20}$$

where

$$V_0(x) = \sup_{0\leq u\leq K} V(x,u,0) = \sup_{0\leq u\leq K} E_x\{\int_0^\tau e^{-\lambda t}u(X_t)dt + Pe^{-\lambda\tau},\} \tag{21}$$

where K is the bound on the dividend rate.

Therefore we can divide our problem into two. First we are going to find $V_0(x)$ and the corresponding optimal control $\tilde{u}(x)$ assuming $Z_0 \equiv 0$. Then we solve problem (20) for $Z_0 \geq 0$ and find $V(x)$.

3.1 Step 1: No initial dividend payment $Z_0(x) \equiv 0$

3.1.1 Stochastic control verification properties

Suppose the manager doesn't pay an initial payment at time zero and performs a specially chosen dividend strategy $u = u(x)$. Let $V_0(x,u)$ be the expected total discounted dividends corresponding to the manager's strategy added together with the liquidation value received upon bankruptcy

$$V_0(x,u) = \mathbf{E}_x\left(\int_0^\tau e^{-\lambda s}u(X_s)ds + Pe^{-\lambda\tau}\right).$$

Assume we found a value function $V_0 = V_0(x)$ and a strategy $\tilde{u} = \tilde{u}(x)$ which we think could be optimal. To prove that a strategy $\tilde{u} = \tilde{u}(x)$ and a value function $V_0 = V_0(x)$ are optimal it is enough to check the standard stochastic control *verification properties*:

(A) For any admissible control $u = u(x)$

$$V_0(x,u) \leq V_0(x), \text{ for all } x \geq 0 \tag{22}$$

(B) The control $\tilde{u} = \tilde{u}(x)$ satisfies

$$V_0(x,\tilde{u}) = V_0(x). \tag{23}$$

We find the optimal control $\tilde{u}$ and the optimal value function $V(x)$ by a dynamic programming approach. Then we check the conditions described above.

3.1.2 Derivation of variational inequalities

Here we will show that the verification properties follow from the variational inequalities described below. This means that for proving optimality it is enough to check these variational inequalities.

Denote by L_1, L_2 the differential operators

$$L_1 = \mu\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2} - \lambda,\ L_2 = -\frac{d}{dx}. \tag{24}$$

Assume there exist such a value function $V_0(\cdot)$ and such a control $\tilde{u}(x)$ that the following variational inequalities hold:

- *Bellman inequality.* For all admissible controls u

$$L_1 V_0(x) + u(x)(L_2 V_0(x) + 1) \leq 0, \text{ for any } x \geq 0. \tag{25}$$

- *Bellman equation.*

$$L_1 V_0(x) + \tilde{u}(x)(L_2 V_0(x) + 1) = 0, \text{ for any } x \geq 0 \tag{26}$$

.

Let us show that the verification properties are satisfied for such a V_0 and $\tilde{u}(\cdot)$.
Apply Ito's formula to $(e^{-\lambda t} V_0(X_t))_{t \geq 0}$

$$e^{-\lambda t} V_0(X_{t \wedge \tau}) = V_0(X_0) + \int_0^{t \wedge \tau} e^{-\lambda s} \left(L_1 V_0(X_s) + u(X_s) L_2 V_0(X_s) \right) ds + \int_0^{t \wedge \tau} e^{-\lambda s} V_0'(X_s) dW_s.$$

Taking the mathematical expectation $\mathbf{E}_x$ given with respect to $X_0 = x$ and using Bellman's inequality (25) we obtain

$$\begin{aligned} V_0(x) &= \mathbf{E}_x e^{-\lambda t} V_0(X_{t \wedge \tau}) - \mathbf{E}_x \int_0^{t \wedge \tau} e^{-\lambda s} \left(L_1 V_0(X_s) + u(X_s) L_2 V_0(X_s) \right) ds - \mathbf{E}_x \int_0^{t \wedge \tau} e^{-\lambda s} V_0'(X_s) dW_s \\ &\geq \mathbf{E}_x e^{-\lambda t} V_0(X_{t \wedge \tau}) + \mathbf{E}_x \int_0^{t \wedge \tau} e^{-\lambda s} u(X_s) ds - \mathbf{E}_x \int_0^{t \wedge \tau} e^{-\lambda s} V_0'(X_s) dW_s \end{aligned} \tag{27}$$

The last stochastic integral in (27) is a martingale. Therefore its mathematical expectation is equal to zero. Also letting $t \to \infty$ we obtain

$$V_0(x) \geq \mathbf{E}_x P e^{-\lambda \tau} + \mathbf{E}_x \int_0^{\tau} e^{-\lambda s} u(X_s) ds = V_0(x, u)$$

It is obvious that property **(A)** is satisfied. Property **(B)** can be checked in the same way but instead of the Bellman inequality we use the Bellman equation (26).

3.1.3 Free boundary problem

It is clear from variational inequalities (25) and (26) that the optimal control $\tilde{u}(\cdot)$ does depend on the sign of $L_2 V_0(\cdot) + 1$ in the following way

$$\tilde{u}(x) = \begin{cases} 0, & L_2 V_0(x) + 1 < 0 \\ K, & L_2 V_0(x) + 1 \geq 0 \end{cases},$$

When $L_2 V_0(x) + 1 = 0$ *switching* of the optimal control happens. A point $\tilde{x}$ such that $L_2 V_0(\tilde{x}) + 1 = 0$ is called a *switching point*.

From the structure of V_0 one may conjecture that

$$\begin{cases} L_2 V_0(x) + 1 < 0, & x < \tilde{x} \\ L_2 V_0(x) + 1 > 0, & x > \tilde{x} \\ L_2 V_0(\tilde{x}) + 1 = 0, & x = \tilde{x}. \end{cases} \tag{28}$$

We won't discuss the conjecture (28) depending on the structure of V_0 in its general form here. This will be the object of another paper. For our purposes it is enough to assume it to be true, to find an explicit form for V_0 and $\tilde{x}$ and then simply check (28). This is enough to prove optimality, as it implies that the variational inequalities, and hence the verification properties are satisfied.

In any case, we conjecture the optimal dividend rate to be

$$\tilde{u}(x) = \begin{cases} 0, & x < \tilde{x} \\ K, & x \geq \tilde{x} \end{cases}.$$

Therefore we can convert the Bellman equation into a free boundary problem. Adding the initial condition[3] $V(0) = P$ we find the value function $V_0(x) = \sup V(x,u,0)$ as a continuous and bounded solution of the following Stephan problem with free boundary $\tilde{x}$:

$$\begin{aligned} L_1V_0 &= 0,\ x \le \tilde{x}, && (29)\\ L_1V_0 + (L_2V_0+1)K &= 0,\ x \ge \tilde{x}, && (30)\\ V_0(0) &= P \qquad \text{(initial condition)}, && (31)\\ L_2V_0(\tilde{x}) + 1 &= 0 \qquad \text{(switching condition)}. && (32)\end{aligned}$$

3.1.4 The value function as a solution of the free boundary problem

Here we solve the free boundary problem (29 – 32). We shall use the notation

$$\begin{aligned} r_{1,2} &= -\mu \pm \sqrt{\mu^2 + 2\lambda} && (33)\\ u_1(x) &= \frac{r_1e^{r_1x} - r_2e^{r_2x}}{r_1 - r_2} && (34)\\ u_2(x) &= \frac{e^{r_1x} - e^{r_2x}}{r_1 - r_2}, && (35)\\ \rho(K) &= K - \mu - \sqrt{(K-\mu)^2 + 2\lambda} < 0, && (36)\\ v(x) &= e^{\rho x} && (37)\\ Q &= Q(K) = \frac{K}{\lambda} + \frac{1}{\rho(K)} = \frac{K}{\lambda} + \frac{1}{K - \mu - \sqrt{(K-\mu)^2 + 2\lambda}}. && (38)\end{aligned}$$

First we solve equation (29) under initial condition (31). Thus V_0 can be represented for $x < \tilde{x}$ as

$$V_0(x) = Pu_1(x) + Au_2(x), \tag{39}$$

where A is a constant.

Note 3.1.1 *We chose independent solutions of (29) $u_1(\cdot)$ and $u_2(\cdot)$ in such a way that they enjoy the following properties*

$$\begin{aligned} u_1(0) &= 1,\ u_1'(0) = r_1 + r_2 = w'(0),\\ u_2(0) &= 0,\ u_2'(0) = 1 = w(0),\end{aligned}$$

Here $w = u_2'u_1 - u_2u_1'$ is the Wronskian of u_2, u_1,

$$w(x) = e^{(r_1+r_2)x} = e^{-2\mu x}. \tag{40}$$

Note 3.1.2 $u_1(x) = \frac{d}{dx}u_2(x)$ *for all* $x \ge 0$.

Secondly we solve equation (30) taking only a bounded solution. Thus V_0 can be represented for $x > \tilde{x}$ as

$$V_0(x) = Bv(x) + \frac{K}{\lambda}, \tag{41}$$

where B is a constant.

Let us now represent constants A and B as functions of a switching point $\tilde{x}$. Applying switching condition(32) we have

$$\begin{aligned} 1 &= V_0'(\tilde{x}) = Bv'(\tilde{x})\\ 1 &= Pu_1'(\tilde{x}) + Au_2'(\tilde{x}) \overset{note 3.1.2}{=} Pu_1'(\tilde{x}) + Au_1(\tilde{x}).\end{aligned}$$

[3] If the firm is bankrupt from the beginning, it gains only the liquidation value P.

Hence

$$B = \frac{1}{v'(\tilde{x})} \tag{42}$$

$$A = \frac{1 - Pu_1'(\tilde{x})}{u_1(\tilde{x})}. \tag{43}$$

Remember now that $V_0(x)$ must be continuous. Therefore matching together two pieces of $V_0(x)$ for $x < \tilde{x}$ and $x \geq \tilde{x}$ we obtain at $\tilde{x}$

$$Pu_1 + \left(\frac{1}{u_2'} - P\frac{u_1'}{u_2'}\right) u_2 = \frac{v}{v'} + \frac{K}{\lambda} \tag{44}$$

Note that $\frac{v(\tilde{x})}{v'(\tilde{x})} = \frac{1}{\rho}$. Thus, rearranging terms in (44) and applying note 3.1.2 we obtain

$$P\frac{w(\tilde{x})}{u_1(\tilde{x})} + \frac{u_2(\tilde{x})}{u_1(\tilde{x})} = Q, \tag{45}$$

We also can rewrite (45) as

$$\frac{-u_2(x) + Qu_1(x)}{w(x)} = P \tag{46}$$

Or in terms of exponents

$$\frac{1 - Qr_2}{r_1 - r_2} e^{-r_1 \tilde{x}} - \frac{1 - Qr_1}{r_1 - r_2} e^{-r_2 \tilde{x}} = P \triangle \tag{47}$$

3.1.5 Existence of the free boundary $\tilde{x}$

Remember now that the expression (46) or equivalent (47) for $\tilde{x}$ were received upon assumption $\tilde{x} > 0$. Therefore we have to find conditions for this to be true. Let us find restrictions on K and P (or in notation of (46) Q and P) for existence of $\tilde{x}$ such that (46) holds. We shall need several lemmas.

Lemma 3.1.1 *(Behaviour of Q) $Q = Q(K)$ defined by (38) is a strictly increasing function (see Fig(3.1.5)). Moreover*

$$Q(0) = \frac{1}{r_2} < 0 \tag{48}$$

$$Q(+\infty) = \frac{\mu}{\lambda} > 0 \tag{49}$$

$$Q(\frac{\lambda}{2\mu}) = 0. \tag{50}$$

Proof. Take the derivative of $Q(x)$. It is easy to see that $Q'(x) > 0$.$\triangle$

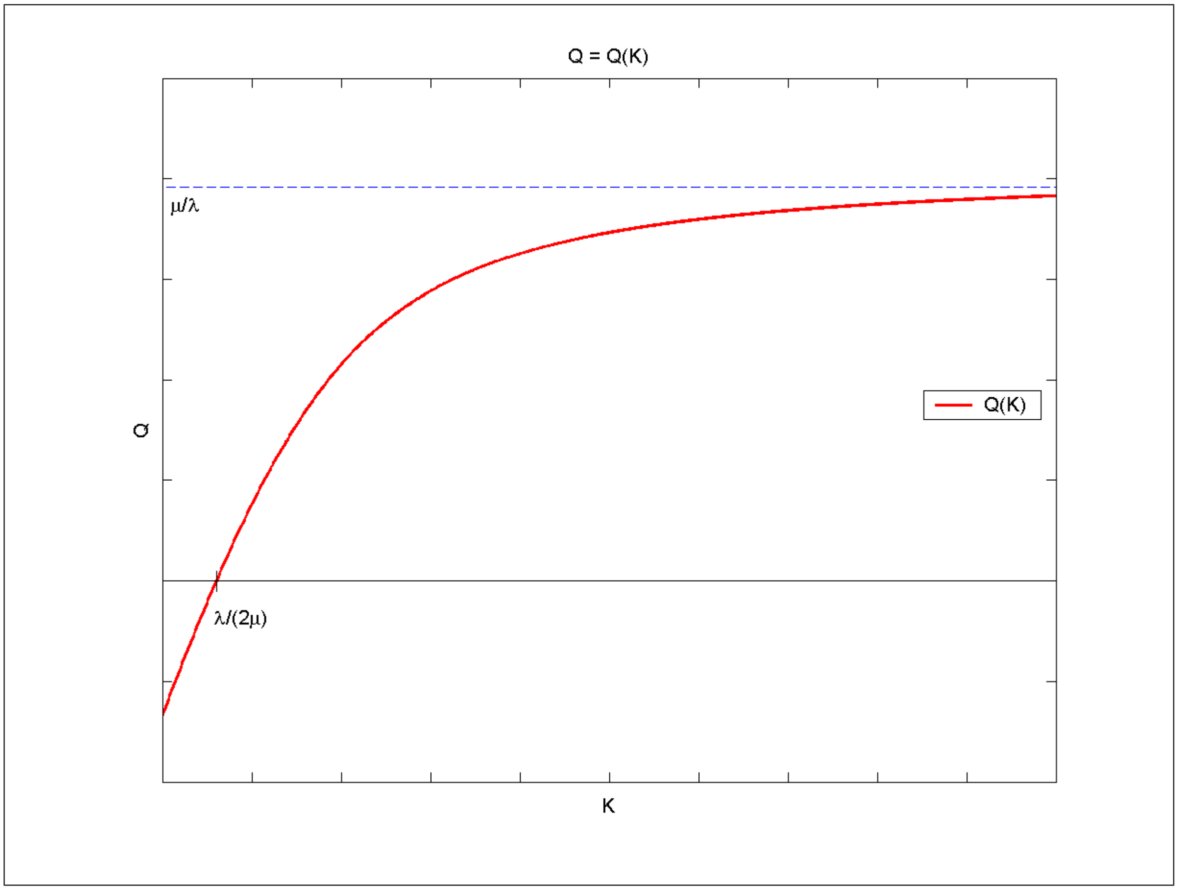

Figure 2. The graph of Q=Q(K).

Let us denote by $f(x, Q)$

$$f(x, Q) = \frac{-u_2(x) + Qu_1(x)}{w(x)}. \tag{51}$$

Now we can formulate the lemma which indicates condition for existence of a switching point.

Lemma 3.1.2 *(When does a switching point exist?)* $f(x, Q) = P, P \geq 0$ *has a solution* $x > 0$ *iff* $Q > 0$ *and* $P < Q$.

From lemma 3.1.1 we have $Q < \mu/\lambda = \frac{r_1+r_2}{r_1 r_2} < \frac{1}{r_1}$. Note that $f(0) = Q$ and $f(x)$ is strictly decreasing function for $x > 0$ since

$$\begin{aligned} f'(x) &= \frac{1}{r_1 - r_2}((-r_2)(Qr_1 - 1)e^{-r_2 x} + r_1(Qr_2 - 1)e^{-r_1 x}) = \\ &= \frac{1}{r_1 - r_2}((-r_2)(Qr_1 - 1)e^{-r_2 x} + (Qr_1 - 1)r_2 e^{-r_1 x} + (r_2 - r_1)e^{-r_1 x}) = \\ &= \frac{1}{r_1 - r_2}[(Qr_1 - 1)(-r_2)(e^{-r_2 x} - e^{-r_1 x}) + (r_2 - r_1)e^{-r_1 x}] < -e^{-r_1 x}. \end{aligned}$$

Therefore $f(x)$ is strictly decreasing function for $x > 0$. It means that $f(x) = P$ has a solution iff $P < f(0) = Q$. △

Corollary 3.1.1 $f(x, Q) = P, P \geq 0$ *has a solution* $x > 0$ *iff* $P < Q$ *(see Fig(3.1.5)).*

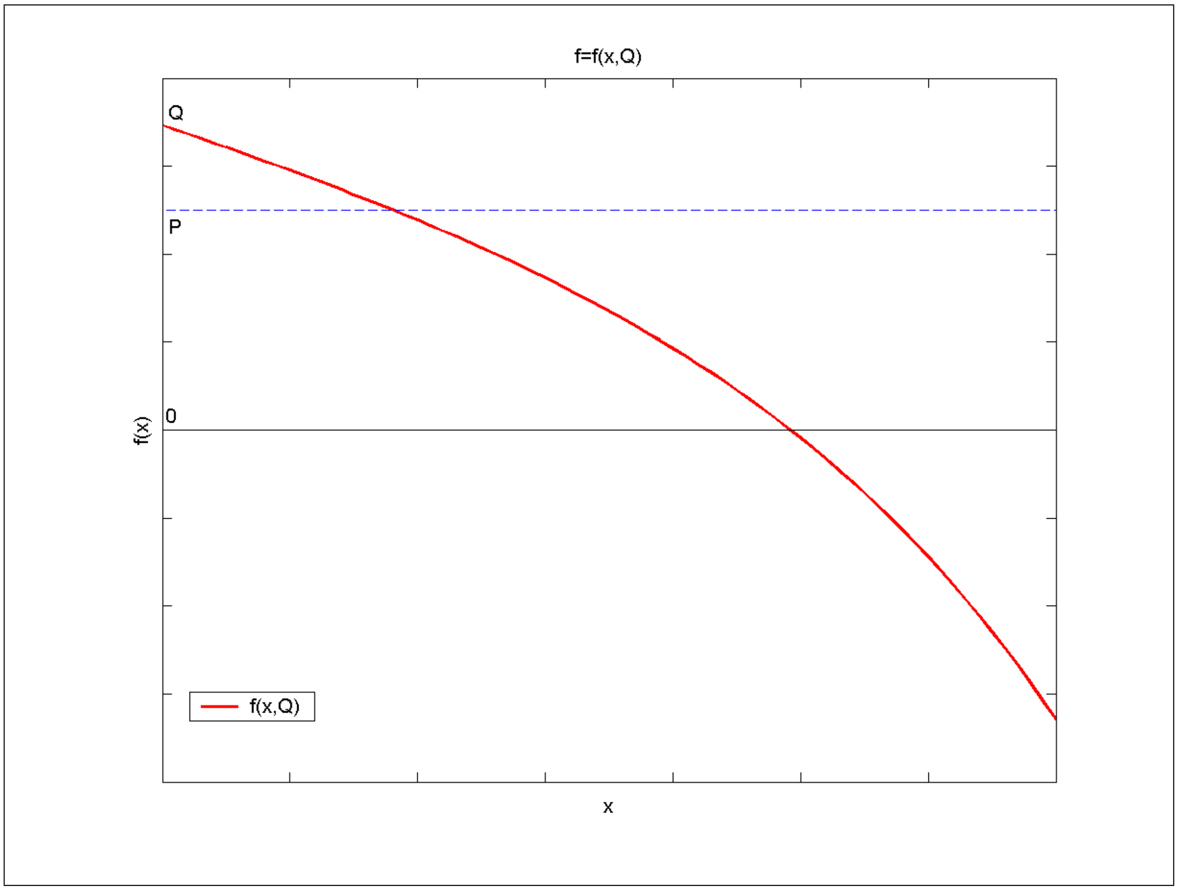

Figure 3. The graph of f(x,Q).

3.1.6 The value function if a free boundary does not exist

Now we know the conjectural solution[4] if $P < Q$. But what happens if $P \geq Q$? From general sense one should put $\tilde{u} = K$[5] and $V_0(x) = \frac{K}{\lambda} + Ce^{\rho x}$, where C is a constant. Applying initial condition $V_0(0) = P$ we get $C = P - \frac{K}{\lambda}$. Consequently for $x \geq \tilde{x}$ we have

$$V_0(x) = \frac{K}{\lambda} + \left(P - \frac{K}{\lambda}\right) e^{\rho x}. \tag{52}$$

3.1.7 Summing up the results

We are now ready to summarize results into the theorem

Theorem 3.1.1 *The value function $V_0 = V_0(x)$ and optimal dividend yield $\tilde{u}(x)$ in case $Z_0 \equiv 0$ are the following*

1) if $P \geq Q(K)$ then $\tilde{u} = K$ and

$$V_0(x) = \frac{K}{\lambda} + \left(P - \frac{K}{\lambda}\right) e^{\rho x}.$$

[4]We still have to check variational inequalities for it

[5]"If liquidation value is quite big then liquidate the firm as fast as you can to get this salvage value".

2) if $P < Q(K)$, then the optimal dividend rate is given by

$$\tilde{u}(x) = \begin{cases} 0, & x < \tilde{x} \\ K, & x \geq \tilde{x} \end{cases},$$

and

$$V_0(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < \tilde{x} \\ \frac{K}{\lambda} - Be^{\rho x}, & x \geq \tilde{x} \end{cases},$$

where $\tilde{x}$ is the root of the equation (47). $u_1(x), u_2(x), v(x)$ are given by (12),(13),(37) respectively and A, B are defined by (43), (42).

3.1.8 Proving optimality

It was shown above that for proving optimality it is enough to check (28). Thus it is enough to prove the following lemma

Lemma 3.1.3 *For $\tilde{V_0} = \tilde{V_0}(x)$ defined by theorem 3.1.1 we have*

$$L_2V_0(x) + 1 < 0, x < \tilde{x}$$
$$L_2V_0(x) + 1 > 0, x > \tilde{x}$$
$$L_2V_0(x) + 1 = 0, x = \tilde{x}.$$

Proof.

Suppose $P < Q$. Then for $x \geq \tilde{x}$ we have

$$L_2V_0 + 1 = 1 - \frac{v'(x)}{v'(\tilde{x})} = 1 - e^{\rho(x-\tilde{x})} > 0, x > 0,$$

since $\rho < 0$.

For $0 \leq x \leq \tilde{x}$ by note 3.1.2 and (43) we have

$$\begin{aligned} L_2V_0(x) + 1 &= 1 - Pu_1'(x) - Au_2'(x) = 1 - Pu_1'(x) - Au_1(x) = \\ &= 1 - Pu_1'(x) - \left(\frac{1}{u_1(\tilde{x})} - P\frac{u_1'(\tilde{x})}{u_1(\tilde{x})}\right) u_1(x) = \\ &= 1 - Pu_1'(\tilde{x}) \left(\frac{u_1(x)}{u_1(\tilde{x})} - \frac{u_1'(x)}{u_1'(\tilde{x})}\right) - \frac{u_1(x)}{u_1(\tilde{x})} < 0, \end{aligned}$$

since $u_1(x)$ is strictly decreasing positive function for $x < \tilde{x}$, and $u_1'(x)$ is strictly increasing function (see Appendix).

Suppose $P \geq Q$. Then

$$L_2V_0(x) + 1 = -\rho\left(P - \frac{K}{\lambda}\right)e^{\rho x} + 1 \geq 0,$$

since the statement $P \geq Q$ is equivalent to $-\rho\left(P - \frac{K}{\lambda}\right) \geq 0$ by definition of $Q\triangle$

Corollary 3.1.2 *For $P < Q$*

$$V_0'(x) > 1, x < \tilde{x}$$
$$V_0'(x) < 1, x > \tilde{x}$$
$$V_0'(x) = 1, x = \tilde{x}.$$

For $P \geq Q$ $V_0'(x) \leq 1$.

3.2 Step 2: Initial dividend payment $Z_0(x) \not\equiv 0$

Assume the manager can provide instantaneous dividend payment $Z_0(x) > 0$. Then by (20)

$$V(x) = \max_{0 \le Z_0 \le x} \{Z_0 + V_0(x - Z_0(x))\}, \tag{53}$$

Define $y = x - Z_0$,

$$\phi(y) = x - y + V_0(y).$$

Then (53) is equivalent to looking for a maximum of $\phi(y)$ over $0 \le y \le x$.

Lemma 3.2.1 *$\phi(y)$ is a strictly increasing function for the case when $P < Q$ and $x < \tilde{x}$.*
$\phi(y)$ is a strictly decreasing function for the case $P \ge Q$ and also for the case $P < Q$, $x < \tilde{x}$.

Proof. Follows from lemma 3.1.3 since $\phi'(y) = -1 + V_0'(y)$. △

Returning to the previous notation we obtain

1. For $P < Q$ and $x < \tilde{x}$ we have $Z_0 = 0$, $V(x) = V_0(x)$.
2. For $P < Q$ and $X \ge \tilde{x}$ we have $Z_0 = x - \tilde{x}$, $V(x) = V_0(\tilde{x}) + x - \tilde{x} = Q + x - \tilde{x}$
3. For $P \ge Q$ we have $Z_0 = x$, $V(x) = V_0(0) + x = P + x$.

Collecting all the results together we see that we have proved the theorem 2.1.1.

4 The case when the dividend process is any nonnegative, nondecreasing and right continuous process.

4.1 The result.

As in [3] this case is the limiting case for the case of bounded dividend rate as $K \to \infty$.

Let $\bar{x}$ be a solution of

$$\frac{r_2^2}{r_1 - r_2} e^{-r_1 x} - \frac{r_1^2}{r_1 - r_2} e^{-r_2 x} = P\lambda. \tag{54}$$

The following theorem suggests the optimal strategy

Theorem 4.1.1 *1) if $P \ge \frac{\mu}{\lambda}$ then the optimal strategy will be to distribute the reserve $X_0 = x$ as dividends immediatly $Z_0 = x$ and go bankrupt $X_t = 0$ for all $t > 0$. $V(x) = P + x$.*

2) if $P < \frac{\mu}{\lambda}$ then

a) if $x > \bar{x}$ we pay dividends $\bar{Z}_0(x) = x - \bar{x}$

b) if $x = \bar{x}$ then the process of dividend payments $\bar{L} = (\bar{L}_t)_{t \ge 0}$ and the reserve process $\tilde{X} = (\tilde{X}_t)_{t \ge 0}$ are the solutions of SDE with reflection $\bar{X}_t = x + \mu t + \sigma W_t - \bar{L}_t$, where $\bar{L} = (\bar{L}_t)_{t \ge 0}$ is a continuous nondecreasing and adapted process with $\bar{L}_0 = 0$ such that $\bar{L}_t = \int_0^t I(\bar{X}_s = \bar{x}) d\bar{L}_s$.

The value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < \bar{x} \\ x - \bar{x} + \frac{\mu}{\lambda}, & x \ge \bar{x} \end{cases},$$

$$A = \frac{1}{u_1(\bar{x})}[1 - Pu_1'(\bar{x})]. \tag{55}$$

4.2 The proof

Again, as in the case of bounded dividend rate we have to check verification properties. This proof is quite standard (see for example [3]). But we include this proof for convenience.

Let $V(x, Z)$ be the expected total discounted dividends corresponded to the manager's strategy added together with the liquidation value received upon bankruptcy

$$V(x, Z) = \mathbf{E}_x \left(\int_0^\tau e^{-\lambda s} dZ_s + Pe^{-\lambda \tau} \right).$$

To prove that the value function V and dividend policy $\tilde{L}$ are optimal we have to check the *verification properties*

(A) For any admissible control Z_t, i.e. any nonnegative nondecreasing cadlag process

$$V(x,Z) \leq V(x), \text{ for all } x \geq 0 \tag{56}$$

(B) The control $\bar{L}$ satisfies

$$V(x,\bar{L}) = V(x). \tag{57}$$

First let us check the property (**B**).

Suppose $x \leq \bar{x}$. Let us show that

$$\mathbf{E}_x\left(\int_0^{\bar{\tau}} e^{-\lambda t} d\bar{L}_t + Pe^{-\lambda\bar{\tau}}\right) = \bar{V}(x), \tag{58}$$

where $\bar{\tau} = \inf(t \geq 0 : \bar{X}_t = 0)$ and the function $\bar{V}(x)$ is defined in theorem 4.1.1. Apply Ito's formulae to a semimartingale $\bar{X}$

$$\begin{aligned} e^{-\lambda(t\wedge\bar{\tau})}\bar{V}(\bar{X}_{t\wedge\bar{\tau}}) &= \bar{V}(x) + \int_0^{t\wedge\bar{\tau}}(-\lambda e^{-\lambda s}\bar{V}(\bar{X}_s))ds + \int_0^{t\wedge\bar{\tau}}\bar{V}'(\bar{X}_s)e^{-\lambda s}d\bar{X}_s \\ &+ \frac{1}{2}\int_0^{t\wedge\bar{\tau}}\sigma^2 e^{-\lambda s}\bar{V}''(\bar{X}_s)ds. \end{aligned}$$

Taking the mathematical expectation with respect to $X_0 = x$ and using the expression for $\bar{X}_t$ from theorem 4.1.1 we obtain

$$\bar{V}(x) = e^{-\lambda(t\wedge\bar{\tau})}\bar{V}(\bar{X}_{t\wedge\bar{\tau}}) - \int_0^{t\wedge\bar{\tau}} e^{-\lambda s}L\bar{V}(\bar{X}_s)ds + \int_0^{t\wedge\bar{\tau}} e^{-\lambda s}\bar{V}'(\bar{X}_s)d\bar{L}_s - \int_0^{t\wedge\bar{\tau}}\sigma e^{-\lambda s}\bar{V}'(\bar{X}_s)dW_s, \tag{59}$$

where $L\bar{V} = -\lambda\bar{V} + \mu\bar{V}' + \bar{V}''$. Note that the stochastic integral in (59) is a martingale and its mathematical expectation is equal to zero. Moreover $L\bar{V} = 0$ by construction. Thus taking into account the smooth pasting condition at $\bar{x}$ $\bar{V}'(\bar{x}) = 1$ and letting t go to infinity we obtain the desired equality (58).

Suppose $x > \bar{x}$. Then dividend process $\bar{Z}_t$ consists of the initial payment $\bar{Z}_0 = x - \bar{x}$ and dividends $\bar{L}_t$

$$\bar{Z}_t = \bar{Z}_0(x)I(x > \bar{x}) + \bar{L}_t.$$

Thus

$$V(x,\bar{Z}) = \mathbf{E}_x\int_0^{\bar{\tau}} e^{-\lambda s}d\bar{Z}_s = \left(x - \bar{x} + \mathbf{E}_{\bar{x}}\int_0^{\bar{\tau}} e^{-\lambda s}d\bar{L}_s\right) + \mathbf{E}_x\int_0^{\bar{\tau}} e^{-\lambda s}d\bar{L}_s\; I(x \leq \bar{x}) = \bar{V}(x).$$

Secondly let us check the property (**A**). By Ito's formula we get

$$\begin{aligned} e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) &= \bar{V}(x) + \int_0^{t\wedge\tau}(-\lambda e^{-\lambda s}\bar{V}(X_s))ds + \int_0^{t\wedge\tau} e^{-\lambda s}\bar{V}'(X_{s-})dX_s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau}\sigma^2 e^{-\lambda s}\bar{V}''(X_s)ds + \sum_{0<s\leq t\wedge\tau} e^{-\lambda s}\bar{V}(X_s) - \bar{V}X_s-) - \bar{V}'(X_{s-}\delta X_s \\ &= \bar{V}(x) - \int_0^{t\wedge\tau} e^{-\lambda s}\bar{V}'(X_{s-}dZ_s + \int_0^{t\wedge\tau} e^{-\lambda s}L\bar{V}(X_s)ds \\ &+ \int_0^{t\wedge\tau}\sigma e^{-\lambda s}\bar{V}'(X_s)dW_s + \sum_{0<s\leq t\wedge\tau} e^{-\lambda s}\bar{V}(X_s) - \bar{V}(X_{s-}) - \bar{V}'(X_{s-})\Delta X_s. \end{aligned}$$

It follows that

$$\begin{aligned}\bar{V}(x) &= \int_0^{t\wedge\tau} e^{-\lambda s} dZ_s - \int_0^{t\wedge\tau} e^{-\lambda s}(1-\bar{V}'(X_{s-}))dZ_s - \int_0^{t\wedge\tau} e^{-\lambda s} L\bar{V}(X_s)ds - \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\bar{V}'(X_s)dW_s \\ &- \sum_{0<s\le t\wedge\tau} e^{-\lambda s}\left(\bar{V}(X_s) - \bar{V}(X_{s-}) - \bar{V}'(X_{s-})\Delta X_s\right) + e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) \qquad (60) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned}$$

Let us study terms of (60)

term2: Since $\bar{V}'(x) \ge 1$

$$I_2 = -\int_0^{t\wedge\tau} e^{-\lambda s}(1-\bar{V}'(X_{s-}))dZ_s \ge 0.$$

term3: For $x < \bar{x}$ we have $L\bar{V}(x) = 0$. And for $x \ge \bar{x}$

$$-L\bar{V}(x) = \lambda\bar{V}(x) - \mu \ge \lambda\bar{V}(\bar{x}) - \mu = -L\bar{V}(\bar{x}) = 0.$$

term4: Since $\bar{V}'(x)$ is bounded we have

$$\mathbf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\bar{V}'(X_s)dW_s = 0$$

term5: For $\alpha < \beta$

$$\bar{V}(\beta) - \bar{V}(\alpha) - \bar{V}'(\beta)(\beta-\alpha) = \int_\alpha^\beta (\bar{V}'(y) - \bar{V}'(\beta))dy \ge 0,$$

since $\bar{V}'(y)$ is a decreasing function. Thus

$$\sum_{0<s\le t\wedge\tau} e^{-\lambda s}\left(\bar{V}(X_s) - \bar{V}(X_{s-}) - \bar{V}'(X_{s-})\Delta X_s\right) \ge 0.$$

term6:

$$\lim_{t\to\infty} \mathbf{E}_x e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) = Pe^{-\lambda\tau}.$$

This implies that taking mathematical expectation $\mathbf{E}_x$ of (60) we obtain the property (**B**).

$$\bar{V}(x) \ge \mathbf{E}_x \int_0^\tau e^{-\lambda s} dZ_s = V(x, Z).$$

This completes the proof.

5 Conclusions and further research

We have found explicit formulas for the optimal dividend flow of a firm with nonnegative liquidation value assuming reserve follows an arithmetic Brownian motion. Similar techniques can be used to study optimal control problems of one-dimensional diffusions, for which Hamilton-Jacobi-Bellman equation can be solved explicitly in terms of special functions (see for more examples Boguslavskaya 2001 [1].) Following the framework of the present paper it is possible to overcome the difficulty of proving the necessity of the so-called "smooth-pasting" condition.

Another interesting problem is to study the case when the dividends can be paid in discrete time with certain fixed transaction cost, i.e. when the dividend process can be represented as

$$Z_t = \sum_{i\ge 0} \xi_i I(T_i \le t), \qquad (61)$$

where $0 = T_0 < T_1 < T_2 < ... < \tau$ are random dividend payment times, $\tau = \inf\{t : X_t \leq 0\}$ and $\xi_0 > 0, \xi_1 > 0, ...$ are the dividend amounts. A non-recoverable cost γ (transaction cost) is associated with each dividend payment. The value function in this case is given by

$$V(x) = \sup \mathbf{E}_x\{\sum_{i\geq 0} e^{-\lambda T_i}(\xi_i - \gamma) + Pe^{-\lambda\tau}\}, \tag{62}$$

where the supremum is taken over all processes (T_i, ξ_i). This problem was posed [3] for the case of zero liquidation value.

6 Appendix

Define by $u_2 = u_2(x)$

$$u_2(x) = \frac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}.$$

Let x^* be a root of the equation

$$u_2''(x) = 0.$$

Then $u_2(x)$, $u_2'(x)$ and $u_2''(x)$ are described by the picture

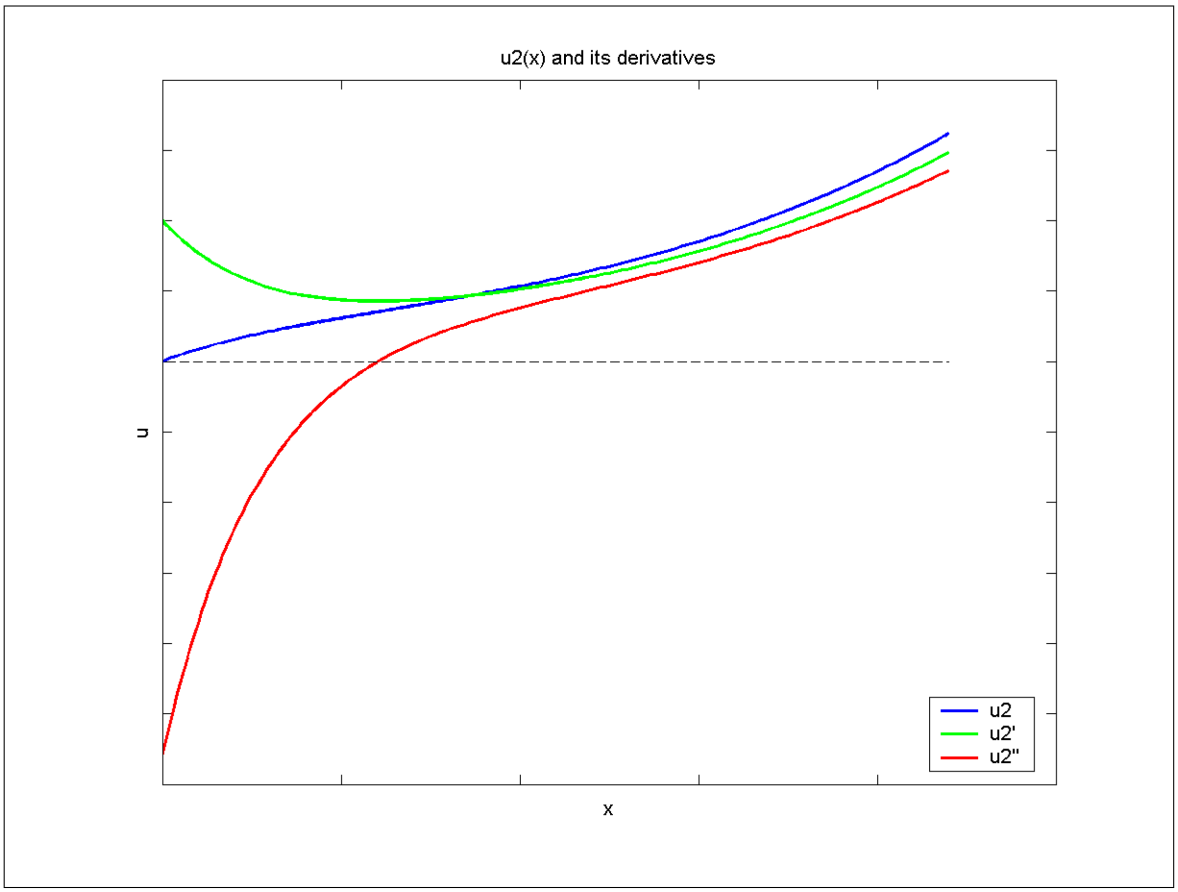

Figure 4. The graph of $u_2(x)$ and its derivatives.

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