On optimization of dividend flow for a company in the presence of liquidation value

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Abstract

We consider a model for a firm whose reserve X_t evolves according to

$$dX_t = \mu dt + \sigma dW_t - dZ_t, \tag{1}$$

where $W = (W_t)_{t \geq 0}$ is a standard Wiener process and μ and σ are positive constants. The control functional Z_t represents the cumulative amount of dividends paid-out up to time t. The major requirement on the control process $Z = (Z_t)_{t \geq 0}$ is that it is nonnegative, nondecreasing and adapted to the filtration. The bankruptcy time τ is defined as $\tau = \inf\{t \geq 0 : X_t \leq 0\}$. It is assumed that the initial reserve $X_0 = x$ is positive and that P is the liquidation value, i.e. the salvage value of the firm's assets at the time of bankruptcy. With λ the constant discount rate, the expected total amount of dividends paid out during the existence of the firm equals

$$V(x, Z) = \mathbf{E}_x\{\int_0^\tau e^{-\lambda t} dZ_t + Pe^{-\lambda\tau}\}. \tag{2}$$

We will present explicit formulae for the optimal admissible process Z, i.e. the control process that maximizes $V(x, Z)$, for the three cases: the case of bounded dividend rates, the case of discrete dividend with transaction cost and for the case when the dividend process is any nonnegative nondecreasing right-continuous process.

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1 Introduction

1.1 Choice of reserve process and structure of dividend process

Radner and Shepp (1996) [RadShepp] have proposed a model for a firm whose reserve X_t in the absence of dividends evolves as an arithmetic Brownian motion[1]

$$dX_t = \mu dt + \sigma dW_t, \tag{3}$$

where μ and σ are positive constants and W_t is a standard Wiener process.

[1] They also showed why geometric Brownian motion is an inappropriate model for the reserve.

The firm'smanager influences the stochastic fluctuations of the company's reserve by choosing the timing and the size of dividend payments. The dynamics of the reserve become

$$dX_t = \mu dt + \sigma dW_t - dZ_t, \tag{4}$$

where the process $Z = (Z_t)_{t\geq 0}$ represents the cumulative amount of dividends paid out up to time t. The process Z_t is assumed to be nonnegative, nondecreasing and right-continuous.

First, we consider a dividend process with bounded dividend rates. In other words, the rate of dividend payments is bounded by a positive constant K. The dividend process Z_t is given by

$$Z_t = Z_0 + \int_0^t u(X_s)ds, \tag{5}$$

where Z_0 is an initial dividend payment, which does not exceed the initial reserve $X_0 = x$, i.e. $0 \leq Z_0 \leq x$. Moreover $u(\cdot)$ is the bounded dividend rate, $0 \leq u(\cdot) \leq K < \infty$. The dividend rate $u = u(\cdot)$ and the initial dividend payment $Z_0 = Z_0(\cdot)$ are assumed to be measurable.

Further, we consider the case when dividend process is given by a multivariate point process $Z_t = (T_i, \xi_i)$

$$Z_t = \sum_{i\geq 0} \xi_i I(T_i < t) \tag{6}$$

and there is a transaction cost $\gamma > 0$ associated with every dividend payment.

Finally, we consider the case where Z_t is assumed to be any nonnegative nondecreasing right-continuous functional. This case is the limit for the previous when $K \to \infty$ or $\gamma \to 0$.

The processes Z_t defined above are called *admissible controls*.

The firm exists from time zero until the first moment τ, at which the cash reserve falls down to zero ($\tau = \inf\{t : X_t \leq 0\}$). The moment τ is called *the bankruptcy time*.

1.2 The value function

The aim of the manager is to maximize expected total discounted dividends paid out during the existence of the firm together with the discounted salvage value at the time of bankruptcy. In other words the *value function* for our optimization problem is given by

$$V(x) = \sup \mathbf{E}_x\{\int_0^\tau e^{-\lambda t} dZ_t + Pe^{-\lambda\tau}\}, \tag{7}$$

where the mathematical expectation is taken conditioned on initial reserve x. Here $P = const$ is the liquidation value, i.e. the salvage value of the firm's assets at the time of bankruptcy, the discount $\lambda = const > 0$ is the positive constant. The supremum is taken over all admissible processes $Z = (Z_t)_{t\geq 0}$.

1.3 Objective

Our aim is to find the value function $V(x)$ and the optimal dividend policy $\tilde{Z}(x)$ given the initial reserve x.

1.4 Normalization

Without loss of generality, we may and will assume $\sigma^2 = 1$ by changing the time scale if necessary. This is achieved on replacing (μ, λ, K) by $(\mu/\sigma^2, \lambda/\sigma^2, K/\sigma^2)$.

1.5 Overview

A survey on the subject may be found in Taksar (1999) [Taksar]. The case of zero liquidation value $P = 0$ was solved by Jeanblanc and Shiryaev (1995) [JeanShir]. Here we present the general solution for the case $P \geq 0$.

2 The main result

2.1 The case of bounded dividend rate

Suppose we are given the initial reserve $x \geq 0$, liquidation value P, discount rate $\lambda > 0$, and an upper bound K on the dividend rate $u(\cdot)$. Let us denote

$$r_1 = -\mu + \sqrt{\mu^2 + 2\lambda} > 0, \tag{8}$$

$$r_2 = \mu + \sqrt{\mu^2 + 2\lambda} > 0, \tag{9}$$

$$\rho = -(K - \mu) + \sqrt{(K - \mu)^2 + 2\lambda} > 0, \tag{10}$$

$$Q = \frac{K}{\lambda} - \frac{1}{\rho}. \tag{11}$$

Define the functions $u_1(\cdot)$, $u_2(\cdot)$ and $v(\cdot)$ on $[0, \infty)$ by

$$u_1(x) = e^{-r_2 x}, \tag{12}$$

$$u_2(x) = e^{r_1 x} - e^{-r_2 x}, \tag{13}$$

$$v(x) = e^{-\rho x}, \tag{14}$$

$$w(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x), \tag{15}$$

$$f(x) = f(x) = P\frac{w(x}{u_2'(x)} + \frac{u_2(x)}{u_2'(x)}. \tag{16}$$

The optimal control process in case of a bounded dividend rate is of the "bang-bang" type and may be described now as follows.

Theorem 2.1.1 *The value function $V = V(x)$ and the optimal dividend strategy in the model for bounded dividend rates are the following.*

1) If $P \geq Q$ then the optimal strategy is to distribute the reserve $X_0 = x$ as dividend immediately, so $Z_0 = x$, and hence to liquidate the company. In this case, $X_t = 0$ for all $t > 0$, the value function is $V(x) = P + x$.

2) If $P < Q$ holds then the optimal instantaneous dividend payment is

$$Z_0(x) = \begin{cases} 0 \text{ if } & x < \tilde{x}, \\ x - \tilde{x} \text{ if } & x \geq \tilde{x}, \end{cases}$$

the optimal dividend rate is given by

$$\tilde{u}(x) = \begin{cases} 0 \text{ if } & x < \tilde{x}, \\ K \text{ if } & x \geq \tilde{x}, \end{cases}$$

and the value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x) \text{ if } & x < \tilde{x}, \\ Q + x - \tilde{x} \text{ if } & x \geq \tilde{x}, \end{cases}$$

with

$$A = \frac{1 + r_2 P e^{-r_2 \tilde{x}}}{r_1 e^{r_1 \tilde{x}} + r_2 e^{-r_2 \tilde{x}}},$$

where $\tilde{x}$ is the unique positive solution of $f(x) = Q$.

This theorem means that whenever the reserve x hits the optimal value $\tilde{x}$, one should start distributing dividends at the maximum possible rate K until the reserve falls back below $\tilde{x}$. This is illustrated in Fig. 2.1.

Note 2.1.1 *In the special case $P = 0$ the statements $P < Q$ and $P \geq Q$ are equivalent to $K > \lambda/(2\mu)$ and $K \leq \lambda/(2\mu)$ respectively (see lemma 3.1.1). Returning to the original parameters these inequalities become $K > \frac{\sigma^2}{2}\frac{\lambda}{\mu}$ and $K \leq \frac{\sigma^2}{2}\frac{\lambda}{\mu}$. These inequalities might have nice economic explanations.*

2.2 The case of discrete dividend

Suppose wee are given the initial reserve $x \geq 0$, liquidation value P, discount rate $\lambda > 0$ and transaction cost $\gamma > 0$. Let r_1, r_2, $u_1(\cdot)$, $u_2(\cdot)$, $f(\cdot)$ be defined as above by (8), (9), (12), (13), (16) respectively. Also define

$$A(x) = \frac{1 - Pu_1'(x)}{u_2'(x)} \tag{17}$$

$$\phi(a, x) = u_2'(x)(A(x) - A(a)). \tag{18}$$

The following theorem gives the optimal strategy for this case.

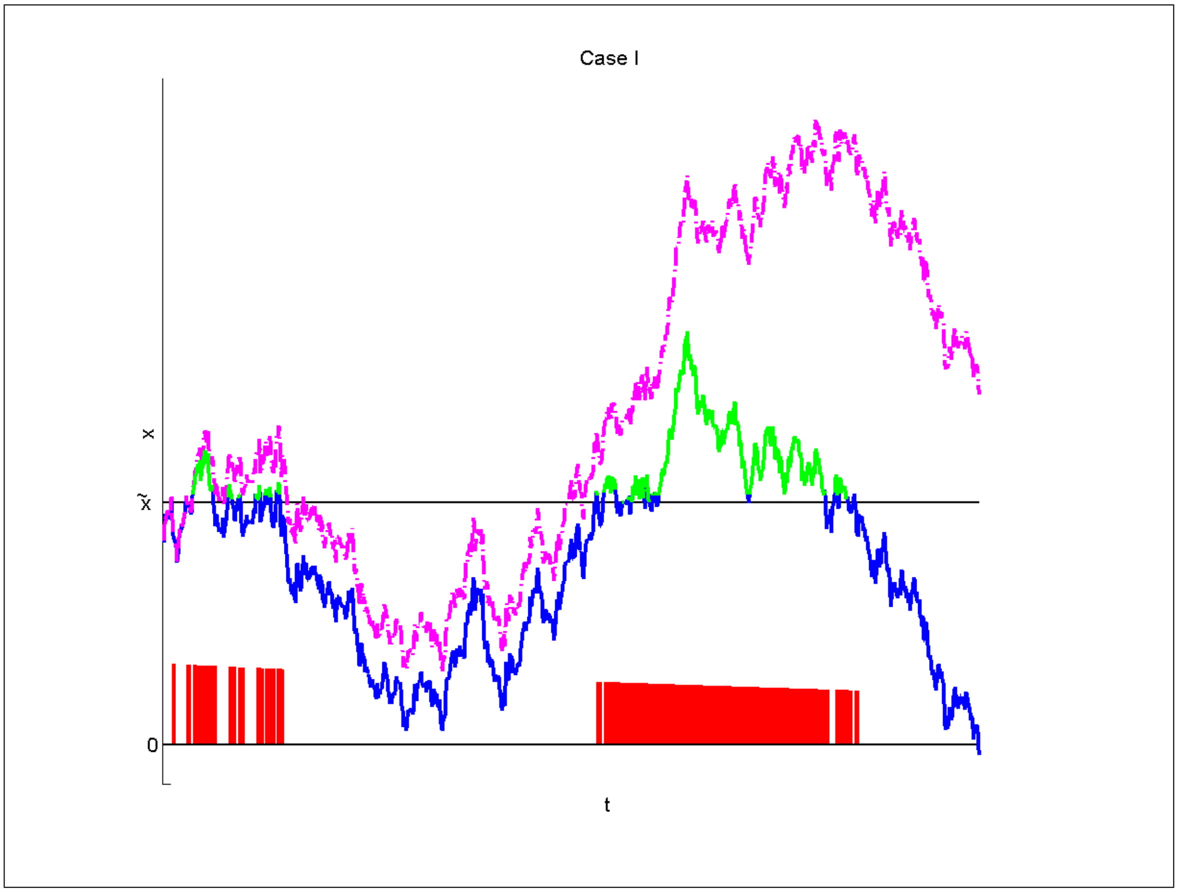

Figure 1: Figure 2.1. An example of the reserve and the dividend payments for the case of bounded dividend rate; *dashed magenta*: an uncontrolled trajectory; *blue:* reserve when the control is off; *green:* reserve when the control is on, *red:* discounted dividends paid.

Theorem 2.2.1 *The value function and the optimal dividend strategy in the model for discrete dividend with transaction cost are the following*

- *In the case of* $P < \mu/\lambda$ *and* $\gamma < \gamma_{max}$ *the value function* $V(\cdot)$ *and optimal dividend process* $Z(c) = (\tilde{\xi}_i, \tilde{T}_i)$ *are given by*
  - *If* $x \geq \tilde{b}$ *then* $\tilde{\xi}_0 = \tilde{b} - \tilde{a}$.
  - *If* $x < \tilde{b}$ *then* $\tilde{T}_0 = 0$, $\tilde{\xi}_0 = 0$ *and* $\tilde{T}_i = \inf(t > T_{i-1} : X_{t-} = \tilde{b})$, $\tilde{\xi}_i = \tilde{b} - \tilde{a}$.

  and the value function

$$V(x) = \begin{cases} Pu_1(x) + A(\tilde{a})u_2(x), & x < \tilde{b} \\ V(\tilde{a}) + x - \tilde{a} - \gamma, & x \geq \tilde{b}, \end{cases}$$

  here pair $(\tilde{a}, \tilde{b})$*, where* $\tilde{b} = b(a)$ *is the unique solution of* $\int_a^{b(a)} \phi(a, y) = \gamma$*, where* $b(a)$ *is such that* $A(a) = A(b(a))$, $b(a) > a$.

- *In the case of* $P < \mu/\lambda$ *and* $\gamma \geq \gamma_{max}$ *or in the case of* $P \geq \mu/\lambda$ *and* $\gamma \geq x$ *the optimal dividend process is*

$$Z(x) = \begin{cases} 0, & x < x^* \\ x, & x \geq x^* \end{cases}$$

  and the value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < x^* \\ x - \gamma + P, & x \geq x^*, \end{cases}$$

  where x^* *is the unique solution of equation* $f(x) = P + x - \gamma$.

- *In the case of* $P \geq \mu/\lambda$ *and* $\gamma < x$ *the optimal strategy is to distribute the reserve as dividend immediately* $Z(x) = x$ *and the value function* $V(x) = P + x - \gamma$.

This means that whenever the reserve x hits a certain level $\tilde{b}$, one should pay $\tilde{b} - \tilde{a}$ in dividend pushing the value of x instantaneously to the value $\tilde{a}$ if only transaction costs are reasonable and the liquidation value is not very big.

2.3 The case when the dividend process is any nonnegative nondecreasing right-continuous process

Suppose as before we are given initial reserve x and liquidation value P, but assume now that the dividend process is any nonnegative, nondecreasing and right-continuous process. This case can be obtained from the discussed above by allowing the bound K on the dividend rate to go to infinity, $K \to \infty$ or by the transaction cost γ go to zero, $\gamma \to 0$. Let $r_1, r_2, u_1(\cdot), u_2(\cdot), f(\cdot)$ be defined as above by (8), (9), (12), (13), (16) respectively. The following theorem presents the optimal strategy for this case.

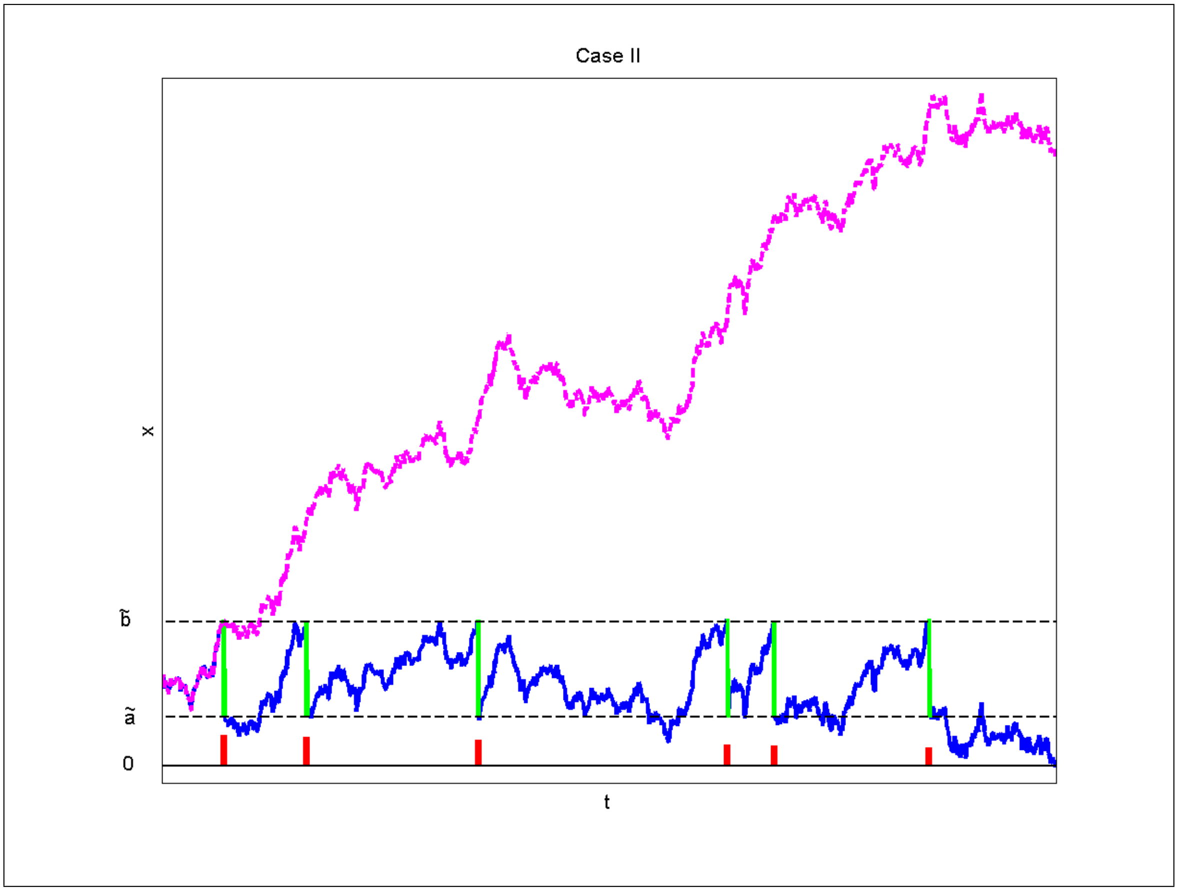

Figure 2: Figure 2.2. An example of the reserve and the dividend payments for the case of discrete dividend with transaction cost; *dashed magenta*: an uncontrolled trajectory; *blue:* reserve when the control is off; *green:* reserve when the control is on, *red:* discounted dividends paid.

Theorem 2.3.1 *The value function $V = V(x)$ and the optimal dividend strategy for the case when the dividend process is any nonnegative nondecreasing right-continuous process are the following.*

- *If $P \geq \frac{\mu}{\lambda}$ then the optimal strategy will be to distribute the reserve $X_0 = x$ as dividends immediately, so $Z_0 = x$, and hence to go bankrupt at the start, so $X_t = 0$ for all $t > 0$. This strategy yields the value function $V(x) = P + x$.*
- *If $P < \frac{\mu}{\lambda}$ then the optimal dividend strategy is the following*

  a) In case of $x > \bar{x}$ dividend $Z_0(x) = x - \bar{x}$ should be paid.

  b) Whenever $x = \bar{x}$ occurs, the process of dividend payments $\bar{L} = (\bar{L}_t)_{t \geq 0}$ and the reserve process $\bar{X} = (\bar{X}_t)_{t \geq 0}$ are the solutions of a SDE with reflection $\bar{X}_t = x + \mu t + W_t - \bar{L}_t$, where $\bar{L} = (\bar{L}_t)_{t \geq 0}$ is a continuous nondecreasing and adapted process with $\bar{L}_0 = 0$ such that $\bar{L}_t = \int_0^t I(\bar{X}_s = \bar{x}) d\bar{L}_s$.

  The value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < \bar{x} \\ x - \bar{x} + \frac{\mu}{\lambda}, & x \geq \bar{x} \end{cases}$$

  with

$$A = \frac{1 + r_2 P e^{-r_2 \bar{x}}}{r_1 e^{r_1 \bar{x}} + r_2 e^{-r_2 \bar{x}}}, \tag{19}$$

  where $\bar{x}$ is the positive solution of equation $f(x) = \mu/\lambda$.

This means[2] that whenever the reserve x hits a certain value $\bar{x}$ from below one should start distributing the dividends so fast that the reserve reflects at $\bar{x}$. The dividends come from the local time spent by the reserve at $\bar{x}$.

3 Obtaining a solution in the case of bounded dividend rate

In the case of a bounded dividend rate the reserve of the company is given by

$$X_t = (x - Z_0) + \int_0^t (\mu - u(X_s)) ds + W_t, \tag{20}$$

where Z_0 is an initial dividend payment, x is the initial reserve, and u is the dividend rate.

[2]remember that this is the limiting case for bounded dividend rate as $K \to \infty$ and for discrete dividend with transaction cost as $\gamma \to 0$.

Let P be the liquidation value. We measure optimality of the dividend policy by a functional of the first dividend lump payment Z_0 and of the dividend rate u, namely

$$\begin{aligned} V(x,u,Z_0) &= Z_0(x) + E_{x-Z_0(x)}\{\int_0^\tau e^{-\lambda t}u(X_t)dt + Pe^{-\lambda\tau}\}, \\ V(x) &= \sup_{0\le u(x)\le K, 0\le Z_0(x)\le x} V(x,u,Z_0). \end{aligned} \tag{21}$$

It is obvious from (21) that

$$V(x) = \max_{0\le Z_0\le x}\{Z_0 + V_0(x - Z_0(x))\}, \tag{22}$$

where

$$V_0(x) = \sup_{0\le u\le K} V(x,u,0) = \sup_{0\le u\le K} E_x\{\int_0^\tau e^{-\lambda t}u(X_t)dt + Pe^{-\lambda\tau}\} \tag{23}$$

with K the bound on the dividend rate.

Therefore we can divide our problem into two subproblems. First we are going to find $V_0(x)$ and the corresponding optimal control $\tilde{u}(x)$ assuming $Z_0 \equiv 0$. Then we use (22) to solve the problem for $Z_0 \ge 0$ and find $V(x)$.

3.1 Step 1: No initial dividend payment, $Z_0 \equiv 0$

3.1.1 Stochastic control verification properties

Suppose the manager is not allowed to pay an initial dividend payment at time zero and uses dividend strategy $u = u(\cdot)$. Let $V_0(x,u)$ be the expected total discounted dividend corresponding to the manager's strategy added together with the discounted liquidation value received upon bankruptcy

$$V_0(x,u) = V(x,u,0) = \mathbf{E}_x\left(\int_0^\tau e^{-\lambda s}u(X_s)ds + Pe^{-\lambda\tau}\right).$$

Assume we find a value function $V_0 = V_0(\cdot)$ and a strategy $\tilde{u} = \tilde{u}(\cdot)$ that we think could be optimal. To prove that the strategy $\tilde{u} = \tilde{u}(\cdot)$ and the value function $V_0 = V_0(\cdot)$ are optimal, it is enough to check the standard stochastic control *verification properties*:

(A) For any admissible control $u = u(\cdot)$

$$V_0(x,u) \le V_0(x), \text{ for all } x \ge 0. \tag{24}$$

(B) The control $\tilde{u} = \tilde{u}(\cdot)$ satisfies

$$V_0(x,\tilde{u}) = V_0(x), \text{ for all } x \ge 0. \tag{25}$$

We find the optimal control $\tilde{u}$ and the optimal value function $V_0(\cdot)$ by a dynamic programming approach. Then we check the conditions described above.

3.1.2 Derivation of variational inequalities

Here we will show that the verification properties follow from the variational inequalities described below. This means that for proving optimality it is enough to check these variational inequalities.

Denote by L_1, L_2 the differential operators

$$L_1 = \mu \frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2} - \lambda, \ L_2 = -\frac{d}{dx}. \tag{26}$$

Assume there exist such a value function $V_0(\cdot)$ and a control $\tilde{u}(x)$ such that the following variational inequalities hold:

- *Bellman inequality.* For all admissible controls u

$$L_1 V_0(x) + u(x)(L_2 V_0(x) + 1) \leq 0, \text{ for any } x \geq 0. \tag{27}$$

- *Bellman equation.*

$$L_1 V_0(x) + \tilde{u}(x)(L_2 V_0(x) + 1) = 0, \text{ for any } x \geq 0. \tag{28}$$

Let us show that the verification properties are satisfied for such a V_0 and $\tilde{u}(\cdot)$.

Apply Îto's formula to $(e^{-\lambda t} V_0(X_t))_{t \geq 0}$

$$e^{-\lambda t} V_0(X_{t\wedge\tau}) = V_0(X_0) + \int_0^{t\wedge\tau} e^{-\lambda s} \left(L_1 V_0(X_s) + u(X_s) L_2 V_0(X_s) \right) ds + \int_0^{t\wedge\tau} e^{-\lambda s} V_0'(X_s) dW_s$$

Taking the mathematical expectation $\mathbf{E}_x$ with respect to $X_0 = x$ and using Bellman's inequality (27) we obtain

$$\begin{aligned} V_0(x) &= \mathbf{E}_x e^{-\lambda t\wedge\tau} V_0(X_{t\wedge\tau}) - \mathbf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} \left[L_1 V_0(X_s) + u(X_s) L_2 V_0(X_s) \right] ds \\ &\quad - \mathbf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} V_0'(X_s) dW_s \\ &\geq \mathbf{E}_x e^{-\lambda t} V_0(X_{t\wedge\tau}) + \mathbf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} u(X_s) ds - \mathbf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} V_0'(X_s) dW_s. \end{aligned} \tag{29}$$

The last stochastic integral in (29) is a martingale. Therefore its mathematical expectation is equal to zero. Also letting $t \to \infty$ we obtain

$$V_0(x) \geq \mathbf{E}_x P e^{-\lambda\tau} + \mathbf{E}_x \int_0^{\tau} e^{-\lambda s} u(X_s) ds = V_0(x, u).$$

It is obvious that property **(A)** is satisfied. Property **(B)** can be checked in the same way but instead of the Bellman inequality we use the Bellman equation (28).

3.1.3 Free boundary problem

It is clear from the variational inequalities (27) and (28) that the optimal control $\tilde{u}(\cdot)$ depends on the sign of $L_2V_0(\cdot)+1$ in the following way

$$\tilde{u}(x) = \begin{cases} 0 \text{ if } & L_2V_0(x)+1<0 \\ K \text{ if } & L_2V_0(x)+1>0. \end{cases}$$

When $L_2V_0(x)+1=0$ the optimal control *switches*. A point $\tilde{x}$ such that $L_2V_0(\tilde{x})+1=0$ is called a *switching point*.

We may conjecture that

$$\begin{cases} L_2V_0(x)+1 < 0 \text{ if } x<\tilde{x} \\ L_2V_0(x)+1 > 0 \text{ if } x>\tilde{x} \\ L_2V_0(\tilde{x})+1 = 0 \text{ if } x=\tilde{x}. \end{cases} \tag{30}$$

For our purposes it is enough to assume (30) to be true, to find an explicit form for $V_0(\cdot)$ and $\tilde{x}$ and then simply to check (30). This is enough to prove optimality, as it implies that the variational inequalities, and hence the verification properties are satisfied.

In any case, we conjecture the optimal dividend rate to be

$$\tilde{u}(x) = \begin{cases} 0 \text{ if } x<\tilde{x} \\ K \text{ if } x\geq\tilde{x}. \end{cases} \tag{31}$$

The implication of (31) allow us to can convert the Bellman equation into a free boundary problem. Adding the initial condition[3] $V(0)=P$ we are looking for the value function $V_0(x)=\sup V(x,u,0)$ as a continuous and bounded solution of the following Stephan problem with free boundary $\tilde{x}$:

$$L_1V_0 = 0,\ x\leq\tilde{x}, \tag{32}$$

$$L_1V_0+(L_2V_0+1)K = 0,\ x\geq\tilde{x}, \tag{33}$$

$$V_0(0) = P \qquad \text{(initial condition)}, \tag{34}$$

$$L_2V_0(\tilde{x})+1 = 0 \qquad \text{(switching condition)}. \tag{35}$$

[3]If the firm is bankrupt from the beginning, it gains only the liquidation value P.

3.1.4 The value function as a solution of the free boundary problem

In this and the next sections we shall solve the free boundary problem (32)–(35). Recall the notation (8)–(14)

$$\begin{aligned} r_1 &= -\mu + \sqrt{\mu^2 + 2\lambda} > 0, \\ r_2 &= \mu + \sqrt{\mu^2 + 2\lambda} > 0, \\ \rho &= -(K-\mu) + \sqrt{(K-\mu)^2 + 2\lambda} > 0, \\ Q &= \frac{K}{\lambda} - \frac{1}{\rho}, \\ u_1(x) &= e^{-r_2 x}, \\ u_2(x) &= e^{r_1 x} - e^{-r_2 x}, \\ v(x) &= e^{-\rho x}. \end{aligned}$$

First we solve equation (32) under the initial condition (34). Thus, $V_0(x)$ can be represented for $x < \tilde{x}$ as

$$V_0(x) = P u_1(x) + A u_2(x), \tag{36}$$

where A is a constant.

Secondly, we solve equation (33) choosing a bounded solution. Thus V_0 can be represented for $x > \tilde{x}$ as

$$V_0(x) = B v(x) + \frac{K}{\lambda}, \tag{37}$$

where B is a constant.

Let us now represent the constants A and B as functions of the switching point $\tilde{x}$. Applying the switching condition (35) we have

$$\begin{aligned} 1 &= V_0'(\tilde{x}) = B v'(\tilde{x}), \\ 1 &= P u_1'(\tilde{x}) + A u_2'(\tilde{x}). \end{aligned}$$

And hence

$$B = \frac{1}{v'(\tilde{x})}, \tag{38}$$

$$A = \frac{1 - P u_1'(\tilde{x})}{u_2'(\tilde{x})}. \tag{39}$$

Recall now that we want $V_0(x)$ to be continuous. Therefore, pasting together the two pieces of $V_0(x)$ for $x < \tilde{x}$ and $x \geq \tilde{x}$ we obtain at $\tilde{x}$

$$P u_1 + \left(\frac{1}{u_2'} - P \frac{u_1'}{u_2'} \right) u_2 = \frac{v}{v'} + \frac{K}{\lambda}. \tag{40}$$

Note that $v(\tilde{x})/v'(\tilde{x}) = -1/\rho$. Thus we obtain

$$P\frac{w(\tilde{x})}{u_2'(\tilde{x})} + \frac{u_2(\tilde{x})}{u_2'(\tilde{x})} = Q, \tag{41}$$

where $w(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$ is the Wronskian of u_1 and u_2.

3.1.5 Existence of free boundary $\tilde{x}$

Recall now that expression (41) for $\tilde{x}$ has been obtained under the assumption $\tilde{x} > 0$. Therefore, we have to find conditions for this to be true. Let us look for restrictions on K and P (or, in the notation of (41), on Q and P) for the existence of $\tilde{x} > 0$ such that (41) holds. We shall need several lemmas. First consider the behaviour of Q as a function of the bound K, see Fig.3.1.5.

Lemma 3.1.1 *Viewed as a function of K, the expression Q defined by (11) is strictly increasing. Moreover, we have*

$$Q(0) = -\frac{1}{r_2} < 0, \tag{42}$$

$$Q(+\infty) = \frac{\mu}{\lambda} > 0, \tag{43}$$

$$Q(\frac{\lambda}{2\mu}) = 0. \tag{44}$$

PROOF. It is easy to see that $Q'(K) = 1/\lambda + \{(\sqrt{(K-\mu)^2 + 2\lambda} - (K - \mu))\sqrt{(K-\mu)^2 + 2\lambda}\}^{-1}$ is positive. The equations (42) – (44) follow by direct computation. △

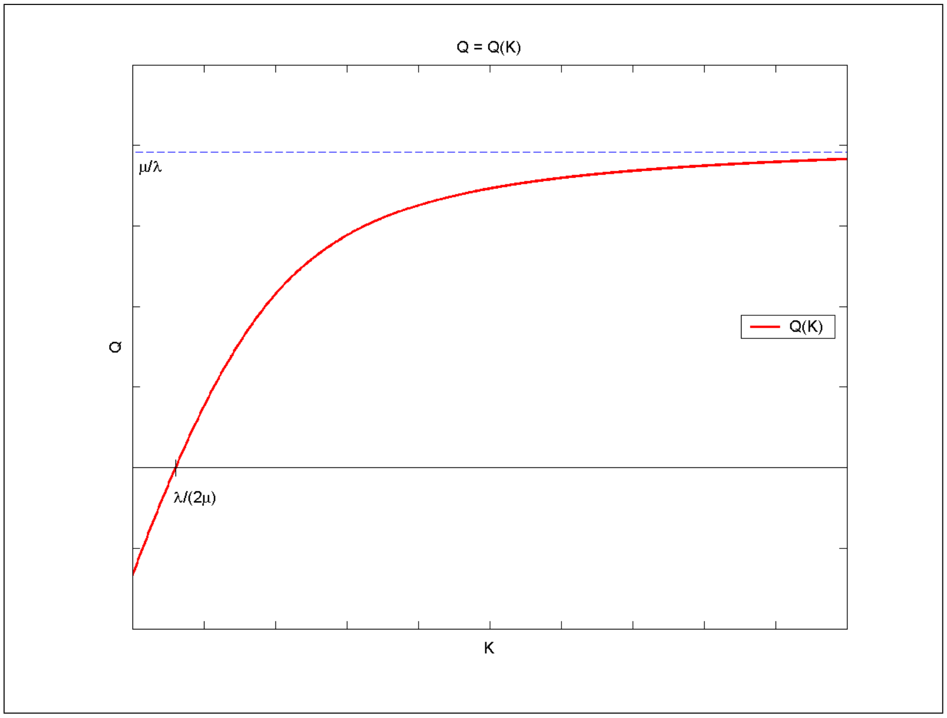

Figure 3.1.5. The graph of Q=Q(K).

Recall

$$f(x) = P\frac{w(x)}{u_2'(x)} + \frac{u_2(x)}{u_2'(x)}. \tag{45}$$

Or in terms of exponentials

$$f(x) = P\frac{(r_1+r_2)e^{(r_1-r_2)x}}{r_1e^{r_1x}+r_2e^{-r_2x}} + \frac{e^{r_1x}-e^{-r_2x}}{r_1e^{r_1x}+r_2e^{-r_2x}}. \tag{46}$$

Now we can formulate a lemma which gives a simple condition for existence of a switching point.

Lemma 3.1.2 *(When does a switching point exist?) The equation $f(x) = Q$ has a solution $x > 0$ iff $P < Q$ holds.*

PROOF. The proof is straightforward after we study the behaviour of $f = f(x)$ (see section 6.2 and fig. 6.2).

- Suppose $P < Q$. By lemma 3.1.1 and results from section 6.2 we have

  $$f(0) = P < Q < \frac{\mu}{\lambda} = \frac{r_2 - r_1}{r_1 r_2} < \frac{1}{r_1} = f(\infty).$$

  Therefore $f(x) = P$ has a unique positive solution when $P < Q$.
- Suppose $P \geq Q$. Again, from section 6.2 we see that for every $x > 0$ $f(x) \geq P > Q$. Therefore there is no solutions to $f(x) = Q$.△

3.1.6 The value function if a free boundary does not exist

Now we know the conjectured solution[4] if $P < Q$. But what happens if $P \geq Q$? By common sense one should put [5] $\tilde{u} = K$ and $V_0(x) = \frac{K}{\lambda} + Ce^{-\rho x}$, where C is a constant. Applying the initial condition $V_0(0) = P$ we get $C = P - \frac{K}{\lambda}$. Consequently, for $x \geq \tilde{x}$ we have

$$V_0(x) = \frac{K}{\lambda} + \left(P - \frac{K}{\lambda}\right) e^{-\rho x}. \tag{47}$$

3.1.7 Summing up the results

We are now ready to summarize our results.

Theorem 3.1.1 *Let $Z_0 \equiv 0$. The value function $V_0(\cdot)$ and the optimal dividend yield $\tilde{u}(\cdot)$ are the following:*

- *If $P \geq Q$ then $\tilde{u} = K$ and*

$$V_0(x) = \frac{K}{\lambda} + \left(P - \frac{K}{\lambda}\right) v(x).$$

- *If $P < Q$, then the optimal dividend rate is given by*

$$\tilde{u}(x) \quad = \quad \begin{cases} 0 \text{ if } & x < \tilde{x} \\ K \text{ if } & x \geq \tilde{x}, \end{cases}$$

  and

$$V_0(x) \quad = \quad \begin{cases} Pu_1(x) + Au_2(x) \text{ if } & x < \tilde{x} \\ \frac{K}{\lambda} + Bv(x) \text{ if } & x \geq \tilde{x}, \end{cases}$$

  where $\tilde{x}$ is the unique root of $f(x) = Q$. The functions $u_1(\cdot), u_2(\cdot)$, $v(\cdot)$, $f(\cdot)$ are given by (12), (13), (14), (16) respectively, and A and B are defined by (39) and (38) respectively.

3.1.8 Proving optimality

It was shown above that for proving optimality it is enough to check (30). Thus it is enough to prove the following lemma.

---

[4] We still have to check the variational inequalities for it.

[5] Intuitively it is clear that the answer should be : "If the liquidation value is quite big, then liquidate the firm as fast as you can to get this salvage value".
Or looking at $f = f(x)$, one notices that in this case $f(x) \geq P \geq Q$. It means that there is no switching boundary, therefore we should distribute dividend immediately.

Lemma 3.1.3 *With $\tilde{x}$ defined as the root of equation $f(x) = Q$, the value function $\tilde{V_0}(\cdot)$ given in theorem 3.1.1 satisfies (30), i.e.*

$$\begin{cases} L_2V_0(x) + 1 < 0, \ x < \tilde{x} \\ L_2V_0(x) + 1 > 0, \ x > \tilde{x} \\ L_2V_0(x) + 1 = 0, \ x = \tilde{x}. \end{cases}$$

Proof.

- Suppose $P < Q$. Then for $x \geq \tilde{x}$ we have

$$L_2V_0(x) + 1 = 1 - \frac{v'(x)}{v'(\tilde{x})} = 1 - e^{-\rho(x-\bar{x})} > 0.$$

  Note, that $\tilde{x}, \bar{x}$, as they are the solutions of $f(x) = Q$ and $f(x) = \mu/\lambda$ respectively and $f(x)$ is an increasing function for $x < \bar{x}$. Then by results form section 6.4 we have for $0 \leq x \leq \tilde{x}$

$$L_2V_0(x) + 1 = 1 - Pu_1'(x) - A(\tilde{x})u_2'(x) = \phi(\tilde{x}, x) < 0.$$

- Suppose $P \geq Q$. Then

$$L_2V_0(x) + 1 = \rho\left(P - \frac{K}{\lambda}\right)e^{-\rho x} + 1 \geq 1 - e^{-\rho x} \geq 0, \ x \geq 0,$$

  since the statement $P \geq Q$ is equivalent to $\rho\left(P - \frac{K}{\lambda}\right) \geq -1$ by definition of Q. $\triangle$

Corollary 3.1.1 *For $P < Q$ we have*

$$\begin{aligned} V_0'(x) > 1, \ x < \tilde{x} \\ V_0'(x) < 1, \ x > \tilde{x} \\ V_0'(x) = 1, \ x = \tilde{x}, \end{aligned}$$

and for $P \geq Q$ we have $V_0'(x) \leq 1$.

3.2 Step 2: Initial dividend payment $Z_0 \not\equiv 0$

Assume the manager is allowed to pay an initial dividend payment $Z_0 > 0$ at time zero . Then by (22) we have

$$V(x) = \max_{0 \leq Z_0 \leq x}\{Z_0 + V_0(x - Z_0)\}, \tag{48}$$

Write $y = x - Z_0$ and $\psi(y) = x - y + V_0(y)$. In order to determine $V(x)$ in (48) we have to maximize $\psi(y)$ over $0 \leq y \leq x$. Since $\Psi'(y) = V_0'(y) - 1 \leq 0$ unless $P < Q$ and

$x < \tilde{x}$ (by Corollary 3.1.1), the maximum of Ψ is attained in $\tilde{x}$ if $P < Q$, and in 0 if $P \geq Q$. Hence returning to the previous notation we obtain
1. For $P < Q$ and $x < \tilde{x}$ we have $Z_0 = 0$, $V(x) = V_0(x)$.
2. For $P < Q$ and $X \geq \tilde{x}$ we have $Z_0 = x - \tilde{x}$, $V(x) = V_0(\tilde{x}) + x - \tilde{x} = Q + x - \tilde{x}$
3. For $P \geq Q$ we have $Z_0 = x$, $V(x) = V_0(0) + x = P + x$.

Collecting all results, we see that we have proved Theorem 2.1.1.

4 Obtaining a solution in the case of discrete dividend with transaction cost.

In the case of discrete dividend with transaction cost the dividend payments are represented by ex-dividend times $0 = T_0 < T_1 < ...$ and by amounts of dividend payments $\xi_0, \xi_1, ...$. A dividend policy (control) is denoted by $\pi = (T_i, \xi_i)_{i\geq 0}$. Dividend process Z_t is given by

$$Z_t = \sum_{i\geq 0} \xi_i I(T_i \leq t \wedge \tau), \tag{49}$$

where τ is the bankruptcy time $\tau = \inf\{s : X_s \leq 0\}$.

The reserve of the company is given by

$$X_t = (x - (\xi_0 - \gamma)) + \mu t + W_t - \sum_{i\geq 1} (\xi_i - \gamma) e^{-\lambda T_i} I(T_i \leq t), \tag{50}$$

where x is the initial reserve, ξ_i are the dividend payments and γ represents the transaction cost.

Let P be the liquidation value. We measure optimality of the dividend policy $\pi = (T_i, \xi_i)_{i\geq 0}$ by a functional of the dividend payments ξ_i made at times T_i (the utility function), namely

$$V(x, \pi) = \mathbf{E}_x\{\sum_{i\geq 0} e^{-\lambda T_i}(\xi_i - \gamma) I(T_i \leq t \wedge \tau) + Pe^{-\lambda\tau}\}, \tag{51}$$

The value function is the supremum of all utility functions over all admissible policies π

$$V(x) = \sup_{\pi} V(x, \pi) = \sup_{(T_i,\xi_i)} \mathbf{E}_x\{\sum_{i\geq 0} e^{-\lambda T_i}(\xi_i - \gamma) I(T_i \leq t \wedge \tau) + Pe^{-\lambda\tau}\}. \tag{52}$$

To find the optimal strategy and the optimal value function we follow the approach of Jeanblanc, Shiryaev (1995) [JeanShir] and use stochastic control verification properties.

4.1 Stochastic control verification properties again

Assume we find a value function $V = V(\cdot)$ and a strategy $\tilde{\pi} = (\tilde{\xi}_i, \tilde{T}_i)$ that we think could be optimal. To prove that the strategy $\tilde{\pi} = (\tilde{\xi}_i, \tilde{T}_i)$ and the value function $V = V(\cdot)$ are optimal, it is enough to check the standard stochastic control *verification properties*:

A1 For any admissible control $\pi = (\xi_i, T_i)$

$$V(x, \pi) \leq V(x), \text{ for all } x \geq 0.$$

B1 The optimal control $\tilde{\pi} = (\tilde{\xi}_i, \tilde{T}_i)$ satisfies

$$V(x, \tilde{\pi}) = V(x), \text{ for all } x \geq 0.$$

We find the optimal control $\tilde{\pi}$ and the optimal value function $V(\cdot)$ by an analogue of stochastic dynamic approach. Then we check the conditions described above.

4.2 An analogue of of variational inequalities

Remind the notation

$$L_1 = \mu \frac{d}{dx} + \frac{1}{2} \frac{d^2}{dx^2} - \lambda, \; L_2 = -\frac{d}{dx}. \tag{53}$$

Here we will show that the verification properties follow from the variational inequalities described below. This means that for proving optimality it is enough to check those **variational inequalities**

I For all admissible controls $\pi = (T_i, \xi_i)$

- **a)** $L_1 V(x) \leq 0$,
- **b)** $\int_{X_{T_i}}^{X_{T_i-}} (1 + L_2 V(u)) du \leq \gamma$, where $\xi_i = X_{T_i-} - X_{T_i}$

for all $x > 0$.

II For control $\tilde{\pi} = (\tilde{T}_i, \tilde{\xi}_i)$

- **a)** $L_1 V(x) = 0$ or $\tau = 0$
- **b)** $\int_{X_{\tilde{T}_i}}^{X_{\tilde{T}_i-}} (1 + L_2 V(u)) du = \gamma$, where $\tilde{\xi}_i = X_{\tilde{T}_i-} - X_{\tilde{T}_i}$.

for all $x > 0$.

Let us show that the verification properties are satisfied for such V and $\tilde{\pi}$.
Apply Îto's formula to $(e^{-\lambda t} V(X_t))_{t \geq 0}$

$$
\begin{aligned}
e^{-\lambda(t\wedge\tau)}V(X_{t\wedge\tau}) &= V(X_0)+\int_0^{t\wedge\tau}(-\lambda e^{-\lambda s}V(X_s))ds \\
&\quad +\int_0^{t\wedge\tau}e^{-\lambda s}V'(X_s)dX_s+\frac{1}{2}\int_0^{t\wedge\tau}e^{-\lambda s}V''(X_s)ds \\
&\quad +\sum_{0\le s\le t\wedge\tau}e^{-\lambda s}\{V(X_s)-V(X_{s-})-\Delta X_s V'(X_{s-})\} \\
&= V(X_0)+\int_0^{t\wedge\tau}e^{-\lambda s}L_1V(X_s)ds+\int_0^{t\wedge\tau}e^{-\lambda s}V'(X_s)dW_s \\
&\quad +\sum_{i\ge 0}e^{-\lambda(T_i\wedge\tau)}\{V(X_{T_i})-V(X_{T_i-})\} \\
&\overset{(\Delta X_{T_i}=-\xi_i)}{=} V(X_0)+\int_0^{t\wedge\tau}e^{-\lambda s}L_1V(X_s)ds \\
&\quad +\int_0^{t\wedge\tau}e^{-\lambda s}V'(X_s)dW_s+\sum_{i\ge 0}e^{-\lambda(T_i\wedge\tau)}\{\xi_i-\gamma\} \\
&\quad +\sum_{i\ge 0}e^{-\lambda(T_i\wedge\tau)}\{V(X_{T_i})-V(X_{T_i-})-(\Delta X_{T_i}+\gamma)\})
\end{aligned}
$$

Taking the mathematical expectation $\mathbf{E}_x$ with respect to $X_0=x$ we obtain

$$
\begin{aligned}
V(x) &= \mathbf{E}_x e^{-\lambda(t\wedge\tau)}V(X_{t\wedge\tau})-\mathbf{E}_x\int_0^{t\wedge\tau}e^{-\lambda s}V'(X_s)dW_s \\
&\quad +\mathbf{E}_x\{\sum_{i\ge 0}e^{-\lambda(T_i\wedge\tau)}(\xi_i-\gamma)\}-\mathbf{E}_x\int_0^{t\wedge\tau}e^{-\lambda s}L_1V(X_s)ds \qquad (54) \\
&\quad -\mathbf{E}_x\{\sum_{i\ge 0}e^{-\lambda T_i}[V(X_{T_i})-V(X_{T_i-})-(\Delta X_{T_i}+\gamma)]I(T_i\le t\wedge\tau))\}
\end{aligned}
$$

The stochastic integral in (54) is a martingale. Therefore its mathematical expectation is equal to zero. Also letting $t\to\infty$ and assuming $V(x)$ bounded, we obtain

$$
\begin{aligned}
V(x) &= \mathbf{E}_x P e^{-\lambda\tau} + \sum_{i\geq 0} e^{-\lambda(T_i\wedge\tau)}\{\xi_i-\gamma\} - \mathbf{E}_x\int_0^\tau e^{-\lambda s}L_1V(X_s)ds \\
&\quad -\sum_{i\geq 0} e^{-\lambda T_i}\{V(X_{T_i})-V(X_{T_i-})-(\Delta X_{T_i}+\gamma)\}I(T_i<\tau)) \\
&= V(x,\pi)-\mathbf{E}_x\int_0^\tau e^{-\lambda s}L_1V(X_s)ds \\
&\quad -\sum_{i\geq 0} e^{-\lambda T_i}\{V(X_{T_i})-V(X_{T_i-})-(\Delta X_{T_i}+\gamma)\}I(T_i<\tau)) \\
&= V(x,\pi)-\mathbf{E}_x\int_0^\tau e^{-\lambda s}L_1V(X_s)ds \\
&\quad -\sum_{i\geq 0} e^{-\lambda T_i}\{\int_{X_{T_i}}^{X_{T_i-}}[1+L_2V(u)]du-\gamma\}I(T_i<\tau) \\
&\overset{(4.2),(4.2)}{\geq} V(x,\pi).
\end{aligned}
$$

It is obvious that property (**A1**) is satisfied. Property (**B1**) can be checked in the same way using (4.2), (4.2).

4.3 Free boundary problem

Looking at the variational inequalities (4.2), (4.2) and (4.2),(4.2) it is natural to propose the following strategy as the optimal control

> *Take two boundaries $a<b$. If the initial value $X_0\geq b$ pay dividends X_0-a. Then pay dividends $b-a$ as soon as process X_t reaches b.*

In other words we think that the following control $\pi=(T_i,\xi_i)$ could be optimal

$$
\begin{aligned}
\xi_0 &= \begin{cases} 0 & \text{if } X_0<b \\ X_0-a & \text{if } X_0\geq b \end{cases}, \\
T_i &= \inf\{t>T_{i-1}: X_{t-}=b\}, \\
X_{T_i} &= a, \\
\xi_i &= -\Delta X_{T_i}=b-a.
\end{aligned}
\tag{55}
$$

The implication of (56) allow us to convert variational inequalities **I, II** from section 4.2 into a free boundaries problem. Adding the initial condition [6] $V(0)=P$ we are looking for value function $V(x)=\sup V(x,\pi)$ as a continuous and bounded solution

[6] If the firm is bankrupt from the beginning, it gains only the liquidation value P.

of the following free boundary problem

$$L_1 V = 0,\ x \le b, \tag{56}$$

$$\int_a^x [1 + L_2 V(u)]du = \gamma,\ x \ge b, \tag{57}$$

$$V(0) = P, \qquad \text{(initial condition)} \tag{58}$$

$$\left.\begin{array}{rcl} L_2 V(a) + 1 & = & 0 \\ L_2 V(b) + 1 & = & 0 \end{array}\right\} \text{(switching conditions).} \tag{59}$$

4.4 The value function as a solution of the free boundary problem

In this and the next sections we shall solve the free boundaries problem (56)–(59)

Recall the notation

$$\begin{array}{rcl} r_1 & = & -\mu + \sqrt{\mu^2 + 2\lambda} > 0, \\ r_2 & = & \mu + \sqrt{\mu^2 + 2\lambda} > 0, \\ u_1(x) & = & e^{-r_2 x}, \\ u_2(x) & = & e^{r_1 x} - e^{-r_2 x}. \end{array}$$

First we solve the equation (56) under the initial condition (58). Thus, $V(x)$ can be represented for $x < b$ as

$$V(x) = P u_1(x) + A u_2(x), \tag{60}$$

where A is a constant. Consequently, we can rewrite (56)–(59) as

$$V(x) = \begin{cases} P u_1(x) + A u_2(x), & \text{if } x \le b, \\ V(a) + x - a - \gamma, & \text{if } x \ge b, \end{cases} \tag{61}$$

$$P u_1'(a) + A u_2'(a) = 1, \tag{62}$$

$$P u_1'(b) + A u_2'(b) = 1. \tag{63}$$

To solve the free boundary problem we have to find a, b, A which satisfy (61) - (63), to prove that the solution is unique and to find conditions under which the solution exists.

4.4.1 The constant A represented as a function of the free boundary.

Let us rewrite constant A as a function of a free boundary by using switching conditions (62) and (63)[7]

[7]See also section 6.3.

$$A(x) = \frac{1 - Pu_1'(x)}{u_2'(x)} = \frac{1 + Pr_2 e^{-r_2 x}}{r_1 e^{r_1 x} + r_2 e^{-r_2 x}} \tag{64}$$

Now, given $A = A(x)$ as a function of free boundary x we can reformulate the problem (61) - (63):

Let $A(x)$ be defined by (64). Find $a, b > 0$ such that $A(a) = A(b)$.

In the following lemma we formulate the conditions under which optimal boundaries $a, b > 0$ exist

Lemma 4.4.1 *Let $A = A(x)$ be defined by (64). There exist $a, b > 0$, $a \neq b$ such that $A(a) = A(b)$ iff $P < \mu/\lambda$.*

PROOF.

Step 1 It can be easily seen that $A(x)$ defined by (64) is a continuous function of x. Then there exist $a, b > 0$ such that $A(a) = A(b)$ iff $A(x)$ has a positive extremum.

Step 2 Here we will see that $A(x)$ has an extremum iff $P < \mu/\lambda$. Indeed, from lemma 6.1.1 $A(x)$ has an extremum at $\bar{x}$ iff $\bar{x}$ satisfies $f(x) = \frac{\mu}{\lambda}$, where $f(x)$ is defined by (45). This equation has a positive solution iff $P < \mu/\lambda$. (The latter statement can be proved exactly in the same way as lemma 3.1.2). $\triangle$

The behaviour of $A = A(x)$ as a function of x when $P < \mu/\lambda$ is shown on Fig. 4.4.1. We denoted by b_{max} the b that corresponds to $a = 0$: $A(b_{max}) = A(0) = (1+Pr_1)/(r_1+r_2)$.

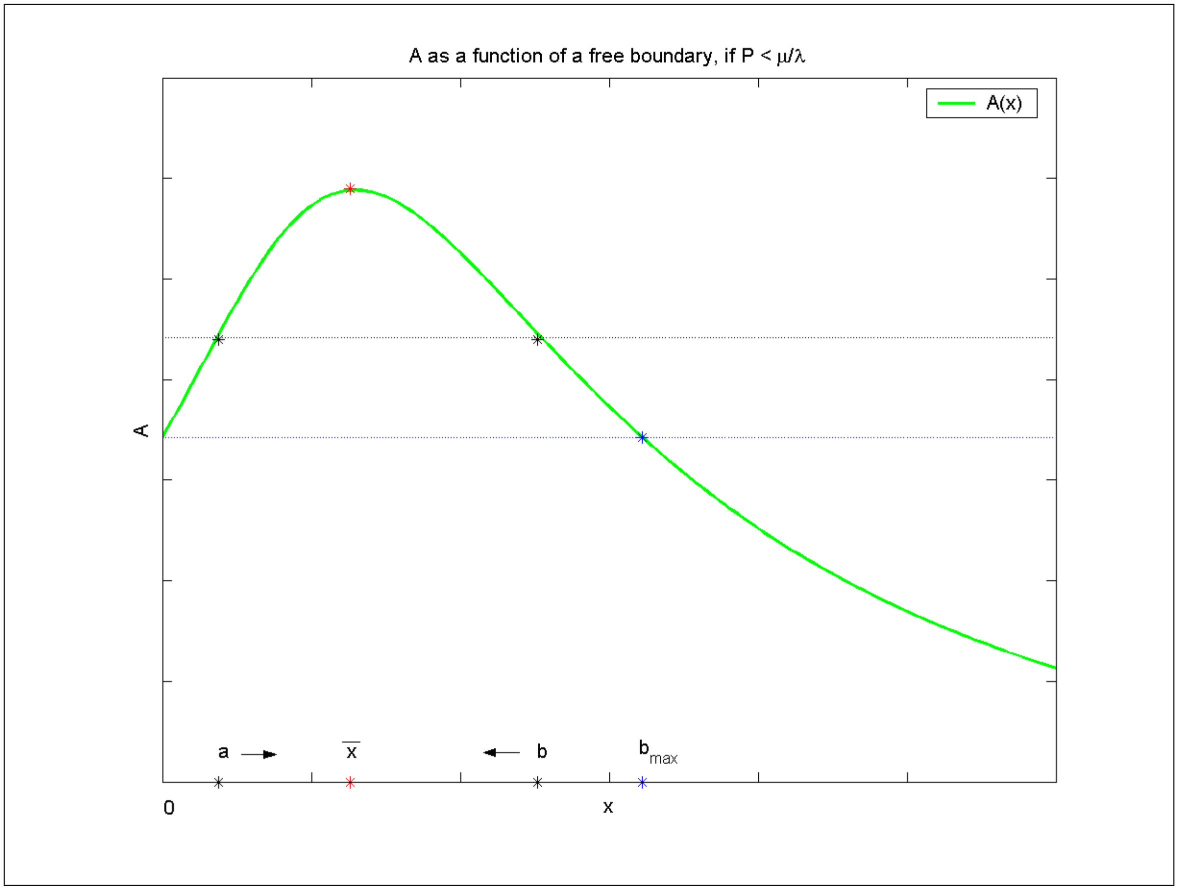

Figure 4.4.1. The graph of A=A(x).

To summarize we conclude

- If $P \geq \mu/\lambda$, then there is no switching boundaries.
- If $P < \mu/\lambda$, then to each lower boundary a, $0 \leq a \leq \bar{x}$, uniquely corresponds an upper boundary $b = b(a)$ such that $A(a) = A(b)$. Therefore all possible switching boundaries a, b can be received by moving boundary a from 0 to $\bar{x}$. The corresponding boundary b moves simultaneously from b_{max} to $\bar{x}$.

4.4.2 The optimal solution and its existence.

Our task now is to find the *optimal* switching boundaries $\tilde{a}$ and $\tilde{b} = b(\tilde{a})$ among all possible switching boundaries $a, b(a)$ from section 4.4.1.

It can be done by studying condition (57) with $x = \tilde{b}$ together with switching conditions (59)

$$\begin{aligned}\int_{\tilde a}^{\tilde b}[1+L_2V(u)]du &= \gamma, && (65)\\ 1+L_2V(\tilde a) &= 0, && (66)\\ 1+L_2V(\tilde b) &= 0, && (67)\end{aligned}$$

where $L_2 = -\frac{d}{dx}$ and $V(x) = Pu_1(x) + A(\tilde a)u_2(x)$.

Denote

$$\begin{aligned}\phi(a,y) &= 1+L_2V(y) = 1-Pu_1'(y)-A(a)u_2'(y)\\ &= u_2'(y)(A(y)-A(a)).\end{aligned}$$

Then we can reformulate (65) - (67) as

$$\int_{\tilde a}^{\tilde b}\phi(\tilde a,y)dy = \gamma,\ \ \phi(\tilde a,y)=0,\ \ \phi(\tilde b,y)=0. \tag{68}$$

Denote $\Gamma(a) = \int_a^{b(a)}\phi(a,y)dy$. Note that $\Gamma(a)$ is a **continuous** function. Moreover $\phi(a,y)\searrow$ as $a\nearrow\bar x$, because $A(a)\nearrow$ as $a\nearrow\bar x$. Also $(b(a)-a)\searrow$ as $a\nearrow\bar x$. Therefore we conclude that $\Gamma(a)$ is a **decreasing** function. It means that starting from $a=0$ and continuously increasing a to $\bar x$, we find uniquely $\tilde a$, $b(\tilde a)$ such that (65) - (67) hold. Of course this holds only if $\gamma < \gamma_{max} = \Gamma(0)$.

$$\gamma_{max} = \Gamma(0) = b_{max} + P + \frac{1}{r_1+r_2}((1-Pr_1)e^{-r_2b_{max}} - (1+Pr_2)e^{r_1b_{max}}). \tag{69}$$

The figure 4.4.2 shows the behaviour of $\phi(a,y)$ as a function of y for different values of a, $0\le a\le\bar x$. The dashed area represents $\gamma=\Gamma(\tilde a)$. Notice that optimal boundaries coincide $\tilde a=\tilde b=\bar x$ when $\gamma=0$. [8]

[8] see the case when dividend process is any nonnegative, nondecreasing and right continuous process.

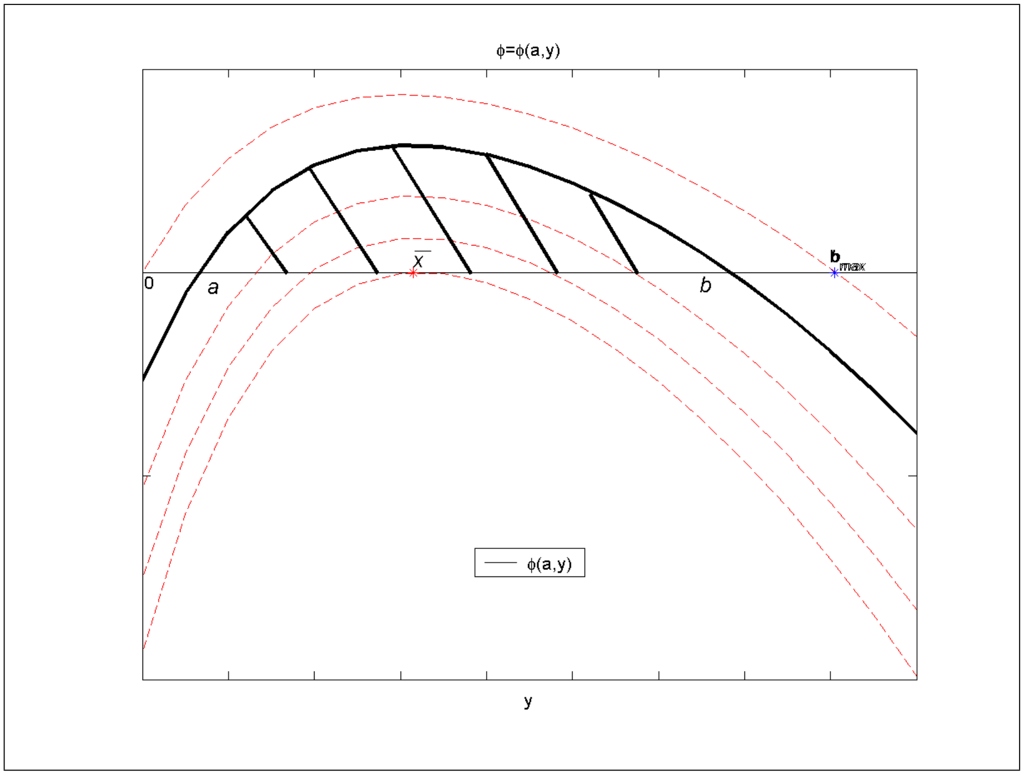

Figure 4.4.2. The graph of $\phi = \phi(a, y)$ for several values of a. The solid line corresponds to a such that the dashed area equals the transaction cost γ.

4.4.3 Summing up the results.

We have found the strategy we think is optimal in the case when $P < \mu/\lambda$ (when switching boundaries exist)[9] and $\gamma < \gamma_{max}$. Let us formulate a theorem about it

Theorem 4.4.1 *In the case of $P < \mu/\lambda$ and $\gamma < \gamma_{max}$ the value function $V(\cdot)$ and optimal dividend process $Z(\cdot) = (\tilde{\xi}_i, \tilde{T}_i)$ are given by*

- *If $x \geq \tilde{b}$ then $\tilde{\xi}_0 = \tilde{b} - \tilde{a}$.*
- *If $x < \tilde{b}$ then $\tilde{T}_0 = 0$, $\tilde{\xi}_0 = 0$ and $\tilde{T}_i = \inf(t > T_{i-1} : X_{t-} = \tilde{b})$, $\tilde{\xi}_i = \tilde{b} - \tilde{a}$.*

The value function is given by

$$V(x) = \begin{cases} Pu_1(x) + A(\tilde{a})u_2(x), & x < \tilde{b} \\ V(\tilde{a}) + x - \tilde{a} - \gamma, & x \geq \tilde{b}, \end{cases}$$

here $\tilde{a}$, $\tilde{b}$ are the solution of $\int_a^{b(a)} \phi(a, y) = \gamma$, where $b(a)$ is such that $A(a) = A(b(a))$, $b(a) > a$.

[9]see lemma 4.4.1

4.4.4 Proving optimality

To prove optimality let us check properties **I** and **II** from section 4.2. Here we have

I **a)**
- Suppose $x < \tilde{b}$. Then $L_1V(x) = 0$,
- Suppose $x \geq \tilde{b}$. We have

$$\begin{aligned} L_1V(x) &= \mu - \lambda V(x) = \lambda\left(\frac{\mu}{\lambda} - V(x)\right) \\ &\leq \lambda\left(\frac{\mu}{\lambda} - V(\tilde{b})\right) \\ &= \lambda\left(f(\bar{x}) - f(\tilde{b})\right) < 0. \end{aligned}$$

b) Note that $\tilde{a} < \bar{x}$. By properties of $\phi(\tilde{a}, x)$ (see section 6.4) we have

$$\int_x^y (1 + L_2V(u))du = \int_x^y \phi(\tilde{a}, u)du \leq \int_{\tilde{a}}^{\tilde{b}} \phi(\tilde{a}, u)du = \gamma.$$

II **a)** $L_1V(x) = 0$ for $x < \tilde{b}$ and $\tau = 0$ for $x \geq \tilde{b}$.

b) By construction

$$\int_{\tilde{a}}^{\tilde{b}} (1 + L_2V(u))du = \gamma.$$

△

4.5 The value function if free boundaries do not exist

In the previous section we have found a strategy of impulse control for the case when transaction cost are not too big ($\gamma < \gamma_{max}$) and the liquidation value is also sufficiently small ($P < \mu/\lambda$). But what to do in the case of large transaction cost or large liquidation value? We discuss it in the following sections.

4.5.1 The case of small liquidation value and large transaction cost

In this case the optimal strategy is not to pay any dividend until the reserve reaches certain level x^*, and as soon as it attends it to distribute all the reserve as dividend.

Theorem 4.5.1 *In the case of $P < \mu/\lambda$ and $\gamma \geq \gamma_{max}$ the optimal dividend process is*

$$Z(x) = \begin{cases} 0, & x < x^* \\ x, & x \geq x^* \end{cases}$$

and the value function is given by

$$V(x) = \begin{cases} Pu_1(x) + Au_2(x), & x < x^* \\ x - \gamma + P, & x \geq x^*, \end{cases}$$

where x^* is the solution of equation $f(x) = P + x - \gamma$.

Proof. To prove optimality let us check properties **I** and **II** from section 4.2. Note, that $x^* > \bar{x}$ by lemma 6.2.1. Here we have

I a)
- Suppose $x < x^*$. Then $L_1 V(x) = 0$.
- Suppose $x \geq x^*$. We have

$$\begin{aligned} L_1 V(x) &= \mu - \lambda V(x) = \lambda \left(\frac{\mu}{\lambda} - V(x) \right) \\ &\leq \lambda \left(\frac{\mu}{\lambda} - V(x^*) \right) \\ &= \lambda \left(f(\bar{x}) - f(x^*) \right) < 0 \end{aligned}$$

b) By properties of $\phi(x^*, x)$ (see section 6.4) we have

$$\int_x^y (1 + L_2 V(u)) du = \int_x^y \phi(x^*, u) du \leq \int_0^{x^*} \phi(x^*, u) du = \gamma$$

II a) $L_1 V(x) = 0$ for $x < \tilde{b}$ and $\tau = 0$ for $x \geq \tilde{b}$.

b) By construction

$$\int_0^{x^*} (1 + L_2 V(u)) du = \gamma.$$

△

4.5.2 The case of large liquidation value

In this case the strategy when you distribute your reserve as soon as you can afford is optimal.

Theorem 4.5.2
- *In the case of $P \geq \mu/\lambda$ and $\gamma < x$ the optimal strategy is to distribute the reserve as dividend immediately.*
- *In the case of $P \geq \mu/\lambda$ and $\gamma \geq x$ the optimal dividend process is given by*

$$Z(x) = \begin{cases} 0, & x < x^{**} \\ x, & x \geq x^{**} \end{cases}$$

and the value function is

$$V(x) = \begin{cases} P u_1(x) + A(x^{**}) u_2(x), & x < x^{**} \\ x - \gamma + P, & x \geq x^{**}, \end{cases}$$

where x^{**} is the solution of equation $f(x) = P + x - \gamma$.

Proof. To prove optimality let us check properties **I** and **II** from section 4.2. Here we have

I a)
- Suppose $x < x^{**}$. Then $L_1 V(x) = 0$,

- Suppose $x \geq x^{**}$. We have

$$\begin{aligned} L_1 V(x) &= \mu - \lambda V(x) = \lambda\left(\frac{\mu}{\lambda} - V(x)\right) \\ &\leq \lambda\left(\frac{\mu}{\lambda} - V(x^{**})\right) \\ < \lambda(P - V(x^{**})) &= \lambda\left(f(0) - f(x^{**})\right) < 0 \end{aligned}$$

b) By properties of $\phi(x^{**}, x)$ (see section 6.4) we have

$$\int_x^y (1 + L_2 V(u))du = \int_x^y \phi(x^{**}, u)du \leq \int_0^{x^{**}} \phi(x^{**}, u)du = \gamma.$$

II **a)** $L_1 V(x) = 0$ for $x < x^{**}$ and $\tau = 0$ for $x \geq x^{**}$.

b) By construction

$$\int_0^{x^{**}} (1 + L_2 V(u))du = \gamma.$$

△

5 The case when the dividend process is any nonnegative, nondecreasing, and right continuous process

5.1 The result.

As in Jeanblanc and Shiryaev (1995) [JeanShir], this case can be considered as limiting for the case of bounded dividend rate as $K \to \infty$ and the case of discrete dividend with transaction cost as $\gamma \to \infty$.

Let us remind the theorem that describes the optimal strategy.

Theorem 5.1.1 *The value function $V = V(x)$ and the optimal dividend strategy in the model when the dividend process is any nonnegative, nondecreasing, and right continuous process are the following*

- *If $P \geq \frac{\mu}{\lambda}$ holds, then the optimal strategy is to distribute the reserve $X_0 = x$ as dividend immediately, so $Z_0 = x$, and hence to liquidate the company at instant $X_t = 0$ for all $t > 0$. $V(x) = P + x$.*
- *if $P < \frac{\mu}{\lambda}$ then*

  a) if $x > \bar{x}$ we pay dividend $\bar{Z}_0(x) = x - \bar{x}$

b) if $x = \bar{x}$ then the process of dividend payments $\bar{L} = (\bar{L}_t)_{t\geq 0}$ and the reserve process $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ are the solutions of SDE with reflection $\bar{X}_t = x + \mu t + W_t - \bar{L}_t$, where $\bar{L} = (\bar{L}_t)_{t\geq 0}$ is a continuous nondecreasing and adapted process with $\bar{L}_0 = 0$ such that $\bar{L}_t = \int_0^t I(\bar{X}_s = \bar{x}) d\bar{L}_s$.

The value function is given by

$$V(x) \quad = \quad \begin{cases} Pu_1(x) + Au_2(x), & x < \bar{x} \\ x - \bar{x} + \frac{\mu}{\lambda}, & x \geq \bar{x} \end{cases}$$

with

$$A = \frac{1 - Pu_1'(\bar{x})}{u_2'(\bar{x})}, \tag{70}$$

where $\bar{x}$ is the solution of $f(x) = \mu/\lambda$.

5.2 The proof

Again, as in the case of bounded dividend rate we have to check the verification properties. This proof is quite standard (see for example [JeanShir]). But we include this proof for convenience.

Let $V(x, Z)$ be the expected total discounted dividend corresponding to the manager's strategy added together with the discounted liquidation value received upon bankruptcy

$$V(x, Z) = \mathbf{E}_x \left(\int_0^\tau e^{-\lambda s} dZ_s + Pe^{-\lambda \tau} \right).$$

To prove that the value function V and dividend policy $\tilde{L}$ are optimal we have to check the *verification properties*

(A) For any admissible control Z_t, i.e. any nonnegative nondecreasing cadlag process

$$V(x, Z) \leq V(x), \text{ for all } x \geq 0 \tag{71}$$

(B) The control $\bar{L}$ satisfies

$$V(x, \bar{L}) = V(x). \tag{72}$$

First let us check the property **(B)**.

Suppose $x \leq \bar{x}$.

Let us show that

$$\mathbf{E}_x \left(\int_0^{\bar{\tau}} e^{-\lambda t} d\bar{L}_t + Pe^{-\lambda \bar{\tau}} \right) = \bar{V}(x), \tag{73}$$

where $\bar{\tau} = \inf\{t \geq 0 : \bar{X}_t = 0\}$ and the function $\bar{V}(x)$ is defined in theorem 2.3.1. Apply Îto's formulae to the semimartingale $\bar{X}$

$$\begin{aligned} e^{-\lambda(t\wedge\bar{\tau})}\bar{V}(\bar{X}_{t\wedge\bar{\tau}}) &= \bar{V}(x) + \int_0^{t\wedge\bar{\tau}}(-\lambda e^{-\lambda s}\bar{V}(\bar{X}_s))ds + \int_0^{t\wedge\bar{\tau}}\bar{V}'(\bar{X}_s)e^{-\lambda s}d\bar{X}_s \\ &+ \frac{1}{2}\int_0^{t\wedge\bar{\tau}} e^{-\lambda s}\bar{V}''(\bar{X}_s)ds. \end{aligned}$$

Using the expression $\bar{X}_t = x + \mu t + W_t - \bar{L}_t$ from theorem 2.3.1 we obtain

$$\begin{aligned} \bar{V}(x) &= e^{-\lambda(t\wedge\bar{\tau})}\bar{V}(\bar{X}_{t\wedge\bar{\tau}}) - \int_0^{t\wedge\bar{\tau}}e^{-\lambda s}L\bar{V}(\bar{X}_s)ds + \int_0^{t\wedge\bar{\tau}}e^{-\lambda s}\bar{V}'(\bar{X}_s)d\bar{L}_s \qquad (74) \\ &- \int_0^{t\wedge\bar{\tau}}e^{-\lambda s}\bar{V}'(\bar{X}_s)dW_s, \end{aligned}$$

where $L\bar{V} = -\lambda\bar{V} + \mu\bar{V}' + \bar{V}''$. Note that the stochastic integral in (74) is a martingale and its mathematical expectation is equal to zero. Moreover $L\bar{V} = 0$ by construction. Thus taking into account the switching condition at $\bar{x}$ (the smooth pasting condition) $\bar{V}'(\bar{x}) = 1$, and letting t go to infinity, we obtain the desired equality (73).

Now suppose $x > \bar{x}$. Then dividend process $\bar{Z}_t$ consists of the initial payment $\bar{Z}_0 = x - \bar{x}$ and dividends $\bar{L}_t$

$$\bar{Z}_t = \bar{Z}_0(x)I(x > \bar{x}) + \bar{L}_t.$$

Thus

$$V(x, \bar{Z}) = \mathbf{E}_x \int_0^{\bar{\tau}} e^{-\lambda s}d\bar{Z}_s = \left(x - \bar{x} + \mathbf{E}_{\bar{x}}\int_0^{\bar{\tau}} e^{-\lambda s}d\bar{L}_s\right) + \mathbf{E}_x\int_0^{\bar{\tau}} e^{-\lambda s}d\bar{L}_s\, I(x \leq \bar{x}) = \bar{V}(x).$$

Secondly let us check the property (**A**). By Îto's formula we get

$$\begin{aligned} e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) &= \bar{V}(x) + \int_0^{t\wedge\tau}(-\lambda e^{-\lambda s}\bar{V}(X_s))ds + \int_0^{t\wedge\tau}e^{-\lambda s}\bar{V}'(X_{s-})dX_s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau}e^{-\lambda s}\bar{V}''(X_s)ds + \sum_{0<s\leq t\wedge\tau}e^{-\lambda s}\{\bar{V}(X_s) - \bar{V}(X_{s-}) - \bar{V}'(X_{s-})\Delta X_s\} \\ &= \bar{V}(x) - \int_0^{t\wedge\tau}e^{-\lambda s}\bar{V}'(X_{s-}dZ_s + \int_0^{t\wedge\tau}e^{-\lambda s}L\bar{V}(X_s)ds \\ &+ \int_0^{t\wedge\tau}e^{-\lambda s}\bar{V}'(X_s)dW_s + \sum_{0<s\leq t\wedge\tau}e^{-\lambda s}\{\bar{V}(X_s) - \bar{V}(X_{s-}) - \bar{V}'(X_{s-})\Delta X_s\}. \end{aligned}$$

It follows that

$$\begin{aligned}
\bar{V}(x) &= \int_0^{t\wedge\tau} e^{-\lambda s} dZ_s - \int_0^{t\wedge\tau} e^{-\lambda s}(1-\bar{V}'(X_{s-}))dZ_s - \int_0^{t\wedge\tau} e^{-\lambda s} L\bar{V}(X_s)ds \\
&\quad - \int_0^{t\wedge\tau} e^{-\lambda s}\bar{V}'(X_s)dW_s \\
&\quad - \sum_{0<s\le t\wedge\tau} e^{-\lambda s}\left(\bar{V}(X_s)-\bar{V}(X_{s-})-\bar{V}'(X_{s-})\Delta X_s\right) \qquad (75) \\
&\quad + e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) \\
&= I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}$$

Let us study terms of (75)

term2: Since $\bar{V}'(x) \ge 1$

$$I_2 = -\int_0^{t\wedge\tau} e^{-\lambda s}(1-\bar{V}'(X_{s-}))dZ_s \ge 0.$$

term3: For $x < \bar{x}$ we have $L\bar{V}(x) = 0$. And for $x \ge \bar{x}$

$$-L\bar{V}(x) = \lambda\bar{V}(x) - \mu \ge \lambda\bar{V}(\bar{x}) - \mu = -L\bar{V}(\bar{x}) = 0.$$

term4: Since $\bar{V}'(x)$ is bounded, we have

$$\mathbf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s}\bar{V}'(X_s)dW_s = 0$$

term5: For $\alpha < \beta$

$$\bar{V}(\beta) - \bar{V}(\alpha) - \bar{V}'(\beta)(\beta-\alpha) = \int_\alpha^\beta (\bar{V}'(y) - \bar{V}'(\beta))dy \ge 0,$$

since $\bar{V}'(y)$ is a decreasing function. Thus

$$\sum_{0<s\le t\wedge\tau} e^{-\lambda s}\left(\bar{V}(X_s)-\bar{V}(X_{s-})-\bar{V}'(X_{s-})\Delta X_s\right) \ge 0.$$

term6:

$$\lim_{t\to\infty} \mathbf{E}_x e^{-\lambda(t\wedge\tau)}\bar{V}(X_{t\wedge\tau}) = Pe^{-\lambda\tau}.$$

This implies that taking mathematical expectation $\mathbf{E}_x$ of (75) we obtain the property (**B**).

$$\bar{V}(x) \ge \mathbf{E}_x\{\int_0^\tau e^{-\lambda s}dZ_s + Pe^{-\lambda\tau}\} = V(x,Z).$$

This completes the proof.

6 Appendix

Recall the notation

$$\begin{aligned}
r_1 &= -\mu + \sqrt{\mu^2 + 2\lambda} > 0, \\
r_2 &= \mu + \sqrt{\mu^2 + 2\lambda} > 0, \\
u_1(x) &= e^{-r_2 x}, \\
u_2(x) &= e^{r_1 x} - e^{-r_2 x}, \\
w(x) &= u_1(x)u_2'(x) - u_1'(x)u_2(x), \\
A(x) &= \frac{1 - Pu_1'(x)}{u_2'(x)}, \\
f(x) &= P\frac{w(x)}{u_2'(x)} + \frac{u_2(x)}{u_2'(x)}
\end{aligned}$$

6.1 Three equivalent ways to find $\bar{x}$.

By $\bar{x}$ we denote the root of equation $f(x) = \mu/\lambda$. As it is shown in lemma 3.1.2 such a root exists if and only if $P < \frac{\mu}{\lambda}$.

Here we present the three equivalent ways to find $\bar{x}$. First observe the following facts

Proposition 6.1.1 *The following statements hold*

$$u_1'(x)u_2''(x) - u_1''(x)u_2'(x) = -r_1 r_2 w(x) \tag{76}$$

$$u_2'' = r_1 r_2 u_2 - (r_2 - r_1)u_2'(x) \tag{77}$$

Proof. Easy to check by direct calculation.

Now we are ready to present the lemma

Lemma 6.1.1 *The following statements are equivalent*

1 *$A(x)$ has the maximum at $\bar{x}$.*

2 *$\bar{x}$ satisfies $f(x) = \mu/\lambda$.*

3 *$\bar{x}$ satisfies $f'(x) = 1$.*

Proof.

$1 \Leftrightarrow 2$ To prove it we take the derivative of $A(x)$

$$\begin{aligned}
A'(x) &= \frac{-Pu_1''(x)u_2'(x) - (1 - Pu_1'(x))u_2''(x)}{(u_2'(x))^2} \\
&= \frac{P(u_1'(x)u_2''(x) - u_1''(x)u_2'(x)) - u_2''(x)}{(u_2'(x))^2} \\
&\overset{prop.6.1.1}{=} \frac{-r_1r_2Pw(x) - r_1r_2u_2 + (r_2 - r_1)u_2'(x)}{(u_2'(x))^2} \\
&= \frac{r_1r_2}{u_2'(x)}\left(\frac{r_2 - r_1}{r_1r_2} - \frac{Pw(x) + u_2(x)}{u_2'(x)}\right) \\
&= \frac{1}{u_2'(x)}\left(\frac{\mu}{\lambda} - f(x)\right).
\end{aligned}$$

$3 \Leftrightarrow 1$

$$\begin{aligned}
f'(x) &= (Pu_1(x) + A(x)u_2(x))' \\
&= Pu_1'(x) + A'(x)u_2(x) + A(x)u_2'(x) \\
&= Pu_1'(x) + A'(x)u_2(x) + 1 - Pu_1'(x) \\
&= 1 + A'(x)u_2(x).
\end{aligned}$$

$\triangle$

Corollary 6.1.1 *$A'(x) > 0$ for $x < \bar{x}$ and $A'(x) < 0$ for $x > \bar{x}$.*

Corollary 6.1.2 *$f'(x) > 1$ for $x < \bar{x}$ and $f'(x) < 1$ for $x > \bar{x}$, $f'(0) = 1$.*

6.2 The behaviour of $f = f(x)$.

Recall

$$\begin{aligned}
f(x) &= P\frac{w(x)}{u_2'(x)} + \frac{u_2(x)}{u_2'(x)} \\
&= P\frac{(r_1 + r_2)e^{(r_1 - r_2)x}}{r_1e^{r_1x} + r_2e^{-r_2x}} + \frac{e^{r_1x} - e^{-r_2x}}{r_1e^{r_1x} + r_2e^{-r_2x}}.
\end{aligned}$$

It can be easily checked by direct calculations that

- $f(0) = P$,
- $f(x)$ is continuous,
- $f(+\infty) = 1/r_1$.

- The closed form expression for the derivative is given by

$$f'(x) = Pr_1 r_2 \frac{(r_1+r_2)e^{(r_1-r_2)x}}{(r_1 e^{r_1 x} + r_2 e^{-r_2 x})^2}\left(\frac{r_1+r_2}{r_1 r_2} - Pu_2(x)\right).$$

The behaviour of $f = f(x)$ is shown by figure 6.2.

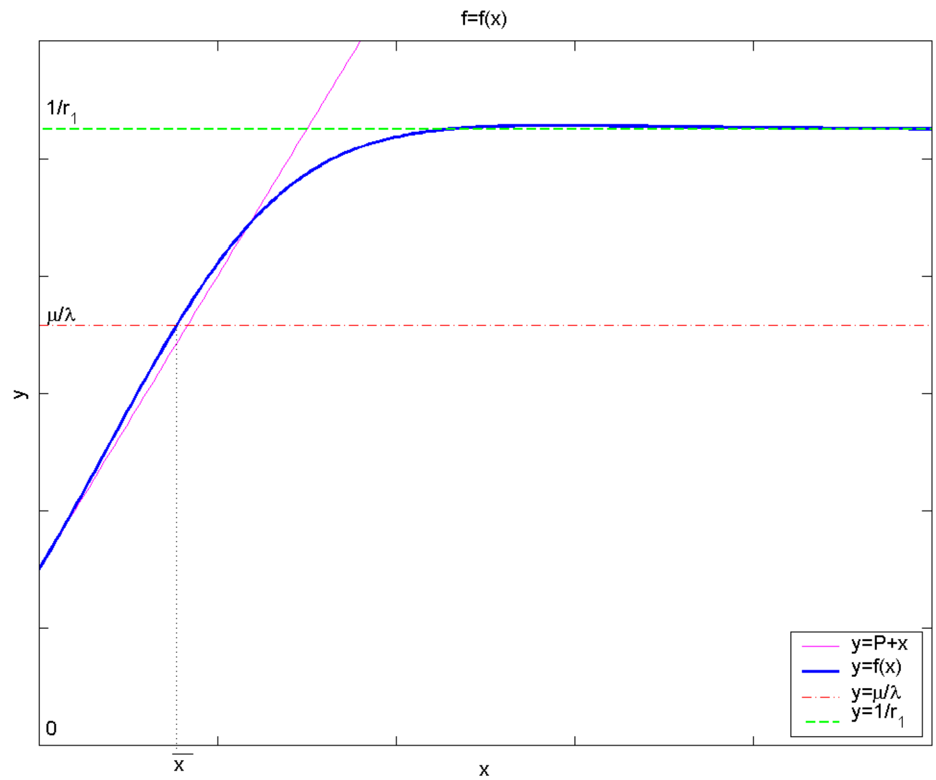

Figure 6.2. Function $f = f(x)$ in the case of reasonable liquidation value $P < \mu/\lambda$.

Lemma 6.2.1 *Assume $P < \mu/\lambda$. Let x_s be a positive solution to $f(x) = P + x - \gamma$. Then this solution is unique and $x_s > \bar{x}$.*

Proof. First we prove the lemma for $\gamma = 0$. From fig. 6.2 we see that for $x < \bar{x}$ $f(x)$ grows quicker than $P + x$ ($f'(0) = 1$, $f'(x) > 1$ for $x < \bar{x}$ from lemma 6.1.2). As a result we have $\mu/\lambda = f(\bar{x}) < f(x_s)$ and, consequently, $\bar{x} < x_s$. Notice, that x_s grows as γ increases. Therefore $x_s > \bar{x}$ for any $\gamma > 0$. The uniqueness of the solution follows from the fact that $f(x)$ grows slower than $P + x$ for $x > \bar{x}$ (lemma 6.1.2) and $f(\bar{x}) > P + \bar{x} - \gamma$. △

6.3 The behaviour of $A = A(x)$.

Recall

$$\begin{aligned} A(x) &= \frac{1 - Pu_1'(x)}{u_2'(x)}, \\ A'(x) &= \frac{1}{u_2'(x)} \left(\frac{\mu}{\lambda} - f(x) \right). \end{aligned}$$

It can be easily checked that

- $A(x)$ is a continuous function
- $A(0) = (1 + Pr_2)/(r_1 + r_2)$.
- Moreover
  - In case $P < \mu/\lambda$

    $$\begin{cases} A(x) \nearrow, & x < \bar{x} \\ A(x) \searrow, & x > \bar{x} \end{cases}$$

    as consequence of $f(x) < \mu/\lambda$ for $x < \bar{x}$ and $f(x) > \mu/\lambda$ for $x > \bar{x}$.
  - In case $P \geq \mu/\lambda$
    $A(x) \searrow$ for $x > 0$ as

    $$A'(x) = \frac{1}{u_2'(x)} \left(\frac{\mu}{\lambda} - f(x) \right) \leq \frac{1}{u_2'(x)} (P - f(x)) = \frac{1}{u_2'(x)} (f(0) - f(x)) < 0.$$

6.4 The behaviour of $\phi(a, y)$.

Denote $\phi(s, x) = u_2'(x)(A(x) - A(s))$. Then we have

- If $P < \mu/\lambda$
  - For any a, b such that $0 \leq a < \bar{x}$, $b > \bar{x}$ and $A(a) = A(b)$ we have

    $$\begin{cases} \phi(a, x) < 0, & x < a \\ \phi(a, x) > 0, & a < x < b \\ \phi(a, x) < 0, & b > x. \end{cases}$$

    as $A(x) < A(a)$ for $0 < x < a$, $A(x) > A(a)$ for $a < x < b$ and $A(x) < A(b)$ for $x > b$.
  - For $b > b_{max}$ (recall that b_{max} was defined as solution to $A(b_{max}) = A(0)$) we have

    $$\begin{cases} \phi(b, x) > 0, & x < b, \\ \phi(b, x) < 0, & x > b, \end{cases}$$

    as $A(x) > A(b)$ for $x < b$ and $A(x) < A(b)$ for $x > b$.

- If $P \geq \mu/\lambda$. For any $y > 0$ we have

$$\begin{cases} \phi(y,x) > 0, & 0 < x < y, \\ \phi(y,x) < 0, & x > y, \end{cases}$$

as for $P \geq \mu/\lambda$ $A(x)$ is a strictly decreasing function.

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